Snow College Mathematics Contest

key

April 3, 2012 **Senior Division: Grades 10-12** **Form: T**

Bubble in the single best choice for each question you choose to answer.

1. Which of the products is/are palindromic (read the same backwards and forwards)?
   (i) 11111×11111
   (ii) 22222×22222
   (iii) 33333×33333
   (iv) 44444×44444

   (A) only (i)
   (B) only (ii) and (iii)
   (C) only (iii) and (iv)
   (D) only (i), (ii), and (iv)
   (E) all of them

   SOLN Look at patterns:

$$\begin{aligned} 1^2 &= 1 \\ 11^2 &= 121 \\ 111^2 &= 12321 \\ 1111^2 &= 1234321 \end{aligned}$$

□

2. If $\sqrt{x} + \sqrt{x+7} = 7$, then what is the value of $2\sqrt{x-5} + \sqrt{x-8}$?
   (A) 5
   (B) 8
   (C) 11
   (D) 14
   (E) $2\sqrt{3}$

   SOLN Squaring both sides gives
   $x + 2\sqrt{x(x+7)} + (x+7) = 49$
   Isolate the square root:
   $\sqrt{x(x+7)} = -x + 21$
   Square again. Then solve for x.
   $x(x+7) = x^2 - 42x + 441$
   $7x = -42x + 441 \implies 49x = 441$
   $\implies x = 9$ $2(2) + 1 = 5$ □

3. The Pauli spin matrices σ_1, σ_2, and σ_3 appear in quantum mechanics. They are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   What is their common determinant?
   (A) 0
   (B) -1
   (C) i
   (D) $-$i
   (E) I

   SOLN $$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

   Note: I, σ_1, σ_2, and σ_3 form a complete basis set for complex 2×2 matrices, so any matrix A can be expressed as $A = c_0 I + c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3$. □

4. $\log_a 2 = 0.356$ and $\log_a 3 = 0.565$ for some mystery base a. Compute $\log_a 12$.
   (A) 0.072
   (B) 0.202
   (C) 0.922
   (D) 1.277
   (E) 1.487

   SOLN $\log_a 12 = \log_a(2^2 \cdot 3) =$
   $\log_a 2^2 + \log_a 3 = 2\log_a 2 + \log_a 3$
   $2(0.356) + 0.565 = 1.277$
   BTW, the mystery base is 7. □

5. For $\triangle$ABC, BC = 13, AC = 14, AB = 15. D is the point on $\overline{\text{AC}}$ such that $\overline{\text{BD}}$ is perpendicular to $\overline{\text{AC}}$. Find the length of $\overline{\text{BD}}$.

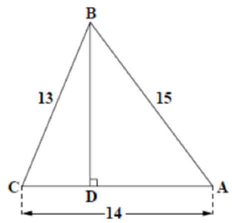
(A) 11

(B) $11\frac{2}{3}$

(C) 12

(D) $12\frac{1}{4}$

(E) $12\frac{1}{3}$



SOLN Let BD = x and CD = y. Since $\triangle$ABD and $\triangle$CBD are right triangles the Pythagorean theorem applies. $x^2+y^2 = 13^2$ and $x^2+(14-y)^2 = 15^2$ Eliminate x^2 and y^2 to get $28y = 140$ so $y = 5$ and $x = 12$. □

6. In all countries but the USA and Canada paper sizes are such that a larger sheet made from two equal sheets of the next smaller size has the same aspect ratio as the smaller sheets. What is that aspect ratio?

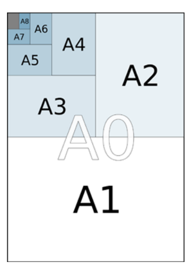
(A) $\sqrt{2} : 1$

(B) $2 : 1$

(C) $3 : 2$

(D) $\frac{1+\sqrt{5}}{2} : 1$

(E) $8 : 5$



SOLN Call x the long side of a big sheet and 1 the short side. Then 1 is the long side of the smaller sheet and $\frac{1}{2}x$ is the short side.

$$\frac{x}{1} = \frac{1}{\frac{1}{2}x} \implies x^2 = 2$$

Note: this makes shrinking two sheets onto one in a copier a cinch. □

7. If $n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot3\cdot2\cdot1$ is the factorial, and the double factorial is

$$n!! = \begin{cases} n\cdot(n-2)\ldots5\cdot3\cdot1 & n>0 \text{ odd} \\ n\cdot(n-2)\ldots6\cdot4\cdot2 & n>0 \text{ even} \\ 1 & n=-1,0 \end{cases}$$

Which of the following statements are true?

(i) $(2n)!! = 2^n\, n!$

(ii) $(2n+1)!!\,2^n\,n! = (2n+1)!$

(iii) $n! = n!!\,(n-1)!!$

(iv) $n!! = (n!)!, \quad n > 2$

(A) only (i) and (ii)

(B) only (ii) and (iii)

(C) only (iii) and (iv)

(D) only (i), (ii), and (iii)

(E) all of them

SOLN Note: $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$

(i) $(2n)!! = (2n)(2n-2)(2n-4)\ldots2 = 2(n)2(n-1)2(n-2)\ldots2(1) = 2^n n!$

(ii) $(2n+1)!!\,(2^n\,n!) = (2\text{n}+1)!!\,(2\text{n})!! = (2\text{n}+1)!$

(ii) $n!!\,(n-1)!! = (n)(n-1)\ldots2\cdot1$

(iv) $(n!)! > n! > n!!$ for $n > 2$ □

8. When 15 is added to a set of ten numbers, the median changes from 6 to 8. Find the median of the new set if 7 replaces 15.

(A) 4

(B) 5

(C) $5\frac{1}{2}$

(D) 6

(E) 7

SOLN Originally, the median, 6, is the average of the 5th and 6th numbers, and the 6th must be 8 (because it is the new median when 15 is added). Replacing 15 with 7 puts the 7 in the middle location (6th). □

9. A box contains two coins: one has heads on both sides; the other is a regular coin. A coin is selected at random and one side is observed to be heads. What is the probability that the other side is also heads?

$\boxed{\text{(A)}}$ 2/3

(B) 1/4

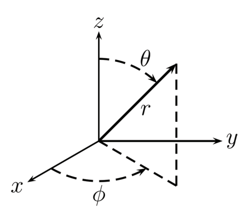
(C) 3/4

(D) 1/3

(E) 5/8

$\boxed{\text{SOLN}}$ Label the two-headed coin "H#1" one side and "H#2" the other. Label the regular coin "H" and "T." You have an equal chance of seeing any of these sides: (a) H; (b) T; (c) H#1; (d) H#2. You see one of the heads; this eliminates (b). Only three possibilities remain: (a) H (with T on the reverse); (c) H#1 (with H#2 on the reverse), and (d) H#2 (with H#1 on the reverse). Marilyn vos Savant's Parade Magazine column on 15 May 2011. □

10. Spherical coordinates (r, θ, ϕ) are defined as shown. Which ranges of variables would cover exactly once all points inside a solid sphere of radius a centered at the origin?



$\boxed{\text{(A)}}$ $0 \le r < a, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi$

(B) $0 \le r < \frac{a}{2}, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$

(C) $0 \le r < a, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{2}$

(D) $0 \le r < a, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le \pi$

(E) $0 \le r < a, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le 2\pi$

$\boxed{\text{SOLN}}$ Some texts reverse the definitions of θ and ϕ. Some replace r with ρ. □

11. Say you place a 25 000-mile-long metal band snugly around the earth's equator. (Assume a smooth spherical earth.) Then you cut the band and splice another 50 feet to it, thus loosening it all around. What is the tallest object that could comfortably fit between the new-length band and the earth?

(A) a DNA molecule

(B) a grain of sand

(C) a golf ball

(D) a small dog

$\boxed{\text{(E)}}$ a tall person

$\boxed{\text{SOLN}}$ The long way is to find the radius of a 25 000-mile-circumference circle and again for a circle of circumference 25 000 miles plus 50 feet.
The short way is to make a straight-line graph of circumference vs. radius for circles.

$$2\pi = \frac{\text{rise}}{\text{run}} = \frac{\Delta C}{\Delta r} \implies$$

$$\Delta r = \frac{\Delta C}{2\pi} = \frac{50\,\text{ft}}{2\pi} \approx 8\,\text{ft}$$

Original circumference is irrelevant. From Marilyn vos Savant's Parade Magazine column on 12 June 2011. □

12. Define $a\#b = ab^2 + a$ for integers $a, b > 0$. If $(a\#b)\#3 = 250$, find $a + b$.

(A) 6

$\boxed{\text{(B)}}$ 7

(C) 8

(D) 9

(E) 10

$\boxed{\text{SOLN}}$ $(ab^2 + a)\#3 = 250$
$(ab^2 + a)3^2 + (ab^2 + a) = 250$
$10ab^2 + 10a = 250$
$a(b^2 + 1) = 25 \implies a = 5, \ b = 2$ □

13. Express the continued fraction expression for x as a simple closed-form number.

$$x = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{\ddots}}}}$$

(A) $\sqrt{5}$

(B) $\frac{1-\sqrt{2}}{2}$

(C) $\frac{1+\sqrt{2}}{2}$

(D) $2 + \frac{\sqrt{2}}{2}$

(E) $\frac{4}{\sqrt{2}}$

SOLN Call the fraction part y. See that the whole equation is equivalent to $x = 2 + y = 2 + \frac{1}{4+y}$. So $y = \frac{1}{4+y}$. Rearrange to get $y^2 + 4y - 1 = 0$ and use the quadratic formula. The two solutions are $y = -2 \pm \sqrt{5}$. $x = 2 + y = \pm\sqrt{5}$ but only the positive one equals the original continued fraction. □

14. Pick any odd number greater than one. Subtract 1 from the square of that odd number. What is the greatest positive integer that must be a divisor of the result?

(A) 2

(B) 3

(C) 4

(D) 8

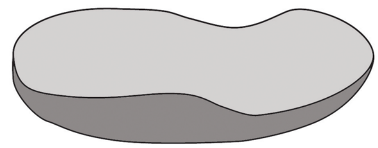
(E) 16

SOLN Odd numbers are $(2n+1)$.

$$(2n+1)^2 - 1 =$$
$$4n^2 + 4n =$$
$$4n(n+1)$$

Either n or $n+1$ is divisible by 2, so $(2n+1)^2 - 1$ is divisible by $4 \cdot 2 = 8$. □

15. A 1/50 scale model of a pond is shown. If the volume of the model is $40\,\text{cm}^3$, then what is the volume of the actual pond?



(A) $5000\,\text{m}^3$

(B) $1.25\,\text{m}^3$

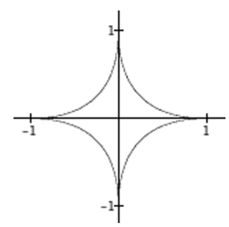
(C) $5\,\text{m}^3$

(D) $500\,\text{m}^3$

(E) $20\,\text{m}^3$

SOLN The 1/50 scale is in each linear dimension; volume is 3-dimensional. $(40\,\text{cm}^3)(50)^3 = 5\,000\,000\,\text{cm}^3$ □

16. The graph of the equation $|x|^n + |y|^n = 1$ with $n = 0.5$ is shown. As n changes, so does the shape of the graph; it is a diamond with vertices on the axes when $n = 1$. What shape does the graph approach as $n \to \infty$?



(A) unit circle

(B) four-leaf clover

(C) square

(D) five-pointed star

(E) two ellipses

SOLN The four points on the axes are always there. When $n = 2$ it is a unit circle. The trend is to push out along the diagonals as n increases. This is a superellipse or Lamé curve. □

17. A machine depreciates by $\frac{1}{5}$ of its current value each year. If it costs \$450 new, what is its value after 2 years?

(A) \$360

(B) \$288 = $(0.8)^2$(\$450)

(C) \$230

(D) \$184

(E) \$147

SOLN Estimate: $(0.8)^2 = (0.64) \approx \frac{2}{3}$
$\frac{2}{3}$ of \$450 is \$300. Pick the closest. □

18. A projectile is fired straight up so that its height in feet above the ground t seconds after firing is $s(t) = -16t^2 + 80t + 96$. Find the maximum height reached and how long it takes to reach that height.

(A) 180 feet; 6 seconds

(B) 180 feet; 3.5 seconds

[(C)] 196 feet; 2.5 seconds

(D) 132 feet; 0.5 seconds

(E) 196 feet; 1 second

[SOLN] $s(0) = 96$ means the projectile starts 96 feet above the ground. Factor: $-16(t+1)(t-6) = 0 \implies t = -1, 6$ when the height is zero. Halfway between (at $t = \frac{5}{2}$) is when it reaches the maximum height. $s\left(\frac{5}{2}\right) = 196$.
A calculus method to maximize s is to set $s'(t) = 0$ and solve for t.
$s'(t) = -32t + 80 = 0 \implies t = \frac{5}{2}$ □

19. A rectangular piece of sheet metal has a length that is 6 in less than twice the width. A 3 in × 3 in square piece is cut from each corner. The sides are then turned up to form an uncovered box of volume 150 in^3. Find the dimensions of the original piece.

(A) $w = 4.5$ in, $l = 9$ in

(B) $w = 8$ in, $l = 10$ in

(C) $w = 10.5$ in, $l = 15$ in

(D) $w = 7$ in, $l = 8$ in

[(E)] $w = 11$ in, $l = 16$ in

[SOLN] $l = (2w - 6)$. Cutting the corners off reduces the width and length of the base of the resulting box by 6 in. $V = l \times w \times h = (2w-12)(w-6)(3) = 150$. $6(w-6)^2 = 150 \implies (w-6)^2 = 25 \implies w - 6 = \pm 5 \implies w = -1$ or 11
Reject -1 on physical grounds. □

20. Jeff paddles his canoe upstream for 3 miles and then returns to his original location. The round-trip takes 2 hours. If the current of the river is 2 mph, how fast does Jeff row his canoe in still water?

(A) 1 mph

(B) 2 mph

(C) 3 mph

[(D)] 4 mph

(E) 5 mph

[SOLN] Say c is his speed in still water.

Dist = Rate × Time

	D (mi)	R ($\frac{\text{mi}}{\text{h}}$)	T (h)
up	3	$c-2$	$\frac{3}{c-2}$
down	3	$c+2$	$\frac{3}{c+2}$
		total:	2

$$\frac{3}{c-2} + \frac{3}{c+2} = 2$$

$$2c^2 - 6c - 8 = 0 = 2(c-4)(c+1)$$

Reject $c = -1$ as unphysical. □

21. Find the value of c so that the equation will have exactly one rational solution.

$$144x^2 - 216x + c = 0$$

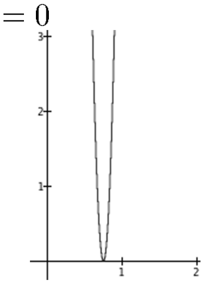
(A) 9

(B) 12

(C) 24

[(D)] 81

(E) 108



[SOLN] Having exactly one rational solution implies that it must factor as $(ax+b)^2 = 0$. Multiply the generic factors to get $a^2x^2 + 2abx + b^2 = 0$. This implies that $a^2 = 144$, $2ab = -216$, and $b^2 = c$. Solving this system gives $a = \pm 12$, $b = \mp 9$, and $c = 81$. □

22. Simplify. $$\left(\frac{4-\mathrm{i}}{1+\mathrm{i}} - \frac{2\mathrm{i}}{2+\mathrm{i}}\right) 4\mathrm{i}$$

(A) $\frac{1}{3} - \frac{3}{2}\mathrm{i}$

(B) $\frac{7}{10} - \frac{21}{10}\mathrm{i}$

(C) $\frac{9}{10} - \frac{27}{10}\mathrm{i}$

[(D)] $\frac{66}{5} + \frac{22}{5}\mathrm{i}$

(E) $11 - 11\mathrm{i}$

SOLN Get a common denominator.

$$\left(\frac{11}{1+3\mathrm{i}}\right) 4\mathrm{i} = \left(\frac{44\mathrm{i}}{1+3\mathrm{i}}\right)\left(\frac{1-3\mathrm{i}}{1-3\mathrm{i}}\right) =$$

$$\frac{132+44\mathrm{i}}{10} = \frac{66}{5} + \frac{22}{5}\mathrm{i}$$ □

23. An n-dimensional hypersphere of radius r has a volume of

$$V_n(r) = r^n \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

where the gamma function is given by

$$\Gamma(n+1) = n\Gamma(n)$$

$$\text{with } \Gamma(1) = 1, \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Find the volume of a 4-D hypersphere of radius 2.

[(A)] $8\pi^2$

(B) $8\pi^4$

(C) $4\pi^2$

(D) $\pi^4/2$

(E) $4\pi^4$

SOLN

$$V_4(2) = 2^4 \frac{\pi^{4/2}}{\Gamma(\frac{4}{2}+1)} = 16\frac{\pi^2}{\Gamma(3)}$$

$$\Gamma(3) = 2\Gamma(2) = (2)(1)\Gamma(1) = 2$$ □

24. What is the value of the following sum?

$$2 + 4 + 6 + 8 + \ldots + 2008 + 2010 + 2012$$

(A) 506520

[(B)] 1013042 $= (2+2012)(1006/2)$

(C) 2026084

(D) 3039126

(E) 4052168

SOLN This is an arithmetic series; sum by adding first and last (or second and penultimate, etc.); then multiply by the number of such pairs. □

25. How many distinct solutions does the equation have in the half-open interval $(0, \pi]$?

$$\sin(x) \cdot \sin(2x) \cdot \sin(3x) \cdot \ldots \cdot \sin(6x) = 0$$

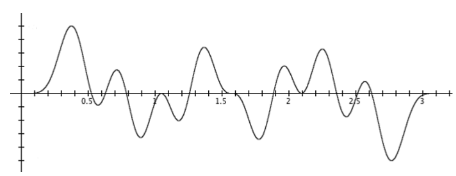
(A) 10

[(B)] 12

(C) 15

(D) 18

(E) 21



SOLN Non-redundant solutions:

$\sin(x) : \pi$

$\sin(2x) : \pi/2$

$\sin(3x) : \pi/3,\ 2\pi/3$

$\sin(4x) : \pi/4,\ 3\pi/4$

$\sin(5x) : \pi/5,\ 2\pi/5,\ 3\pi/5,\ 4\pi/5$

$\sin(6x) : \pi/6,\ 5\pi/6$ □

26. Which is an *anti*derivative of $\sin^2\theta$?

(A) $2\sin\theta\cos\theta$

[(B)] $\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta$

(C) $\frac{1}{3}\sin^3\theta$

(D) $\frac{1}{2}\theta + \frac{1}{4}\cos(2\theta)$

(E) $-\frac{1}{4}\sin(2\theta)$

SOLN Use two trig identities. First use $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$. Then integrate to get $\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)$. The next identity is $\sin(2\theta) = 2\sin\theta\cos\theta$. □

27. What is the coefficient of the x^7y^3 term in the expansion of $(x+y)^{10}$?

- **(A)** 120
- (B) 210
- (C) 720
- (D) 360
- (E) 240

SOLN Brute force works but is slow. Pascal's triangle is a little quicker. The Binomial theorem says the coefficient of the $x^k y^{n-k}$ term is ${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$. ${}_{10}C_7 = \frac{10!}{7!3!} = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}$ □

28. What is the *range* of $f(x) = \frac{3}{1-e^{2x}}$?

- (A) $(0,3)$
- **(B)** $(-\infty,0)\cup(3,\infty)$
- (C) $(3,\infty)$
- (D) $(-\infty,-3)\cup(0,\infty)$
- (E) $f(x)\neq 0,3$

SOLN If $y = \frac{3}{1-e^{2x}}$ $(x \neq 0)$, then $y - ye^{2x} = 3 \implies e^{2x} = \frac{y-3}{y}$. Since the range of e^{2x} is $(0,\infty)$, the problem is reduced to solving $\frac{y-3}{y} > 0$.
Alt. Soln.: Domain is $x \neq 0$.
If $x > 0$ then $1 - e^{2x} < 0$ so $\frac{3}{1-e^{2x}} < 0$.
If $x < 0$ then $1 - e^{2x} > 0$ so $\frac{3}{1-e^{2x}} > 3$.
□

29. The sum of two numbers is 10; their product is 20. Find the sum of their reciprocals.

- (A) $\frac{1}{10}$
- **(B)** $\frac{1}{2}$
- (C) 1
- (D) 2
- (E) 4

SOLN Long way: equations $x + y = 10$ and $xy = 20$ lead to the quadratic formula and rationalizing a denominator.

Short way: $\frac{1}{x}+\frac{1}{y} = \frac{x+y}{xy} = \frac{10}{20}$ □

30. If the margin M (defined as selling price minus cost) made on an article costing C dollars and selling for S dollars is $M = \frac{1}{n}C$, then find the margin in terms of S.

- (A) $M = \frac{1}{n-1}S$
- (B) $M = \frac{1}{n}S$
- (C) $M = \frac{n}{n+1}S$
- **(D)** $M = \frac{1}{n+1}S$
- (E) $M = \frac{n}{n-1}S$

SOLN $M = \frac{1}{n}C = \frac{1}{n}(S-M) \Rightarrow$
$M + \frac{1}{n}M = \frac{1}{n}S \Rightarrow M(\frac{n+1}{n}) = \frac{1}{n}S$ □

31. Find the solution set.

$$\frac{7}{m+4} - \frac{6}{m-4} = \frac{-56}{m^2-16}$$

- (A) $\{-4\}$
- (B) $\{7\}$
- (C) $\{4\}$
- **(D)** $\emptyset$
- (E) {all real numbers}

SOLN Multiply each term by the common denominator $(m+4)(m-4)$. $7(m-4)-6(m+4) = -56 \Rightarrow m = -4$
But -4 is not in the domain. □

32. A student on vacation for d days observed that (1) it rained 7 times, morning or afternoon, (2) when it rained in the afternoon it was clear in the morning, (3) there were 5 clear afternoons, and (4) there were 6 clear mornings. What is d?

- (A) 7
- **(B)** 9
- (C) 10
- (D) 11
- (E) 12

	rainy AM	clear AM
rainy PM	a	b
clear PM	c	e

SOLN $d = a+b+c+e$ $a+b+c=7$
$a=0$ $c+e=5$ $b+e=6$ $e=2$
Alt. Soln: a = afternoon rains; m = morning rains. $a+m=7$ $d-5=a$
$d-6=m$ $2d-11=a+m=7$ □

33. A right circular cone has for its base a circle having the same radius as a given sphere. The volume of the cone is one-half that of the sphere. What is the ratio of the altitude of the cone to the radius of the base?

(A) 1/1

(B) 1/2

(C) 2/3

(D) 2/1

(E) $\sqrt{5/4}$

SOLN

$$V_{\text{cone}} = \frac{1}{2} V_{\text{sphere}}$$

$$\frac{1}{3}\pi r^2 h = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$$

$$r^2 h = 2r^3$$

$$\frac{h}{r} = 2$$

□

34. Assume the following 3 statements are true:

- All teenagers are human.
- All students are human.
- Some students think.

Which of the following are logical consequences of the above statements?

(i) All teenagers are students.

(ii) Some humans think.

(iii) No teenagers think.

(iv) Some humans who think are not students.

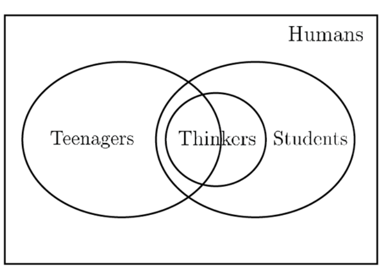
(A) (ii)

(B) (iv)

(C) (ii), (iii)

(D) (ii), (iv)

(E) (i), (ii)



SOLN See the Venn diagram. □

35. Find the 100th digit after the decimal point in $0.\overline{341729}$.

(A) 1

(B) 2

(C) 3

(D) 4

(E) 7

SOLN 6 divides into 100 16 times with remainder 4, so the 100th digit is the 4th digit of the repeating pattern. □

36. My pet rabbit, Cotton, can hop up one step at a time or two steps at a time. The stairs in my house have ten steps. How many ways can Cotton get up my stairs?

(A) 20

(B) 32

(C) 89 = 1 + 15 + 35 + 28 + 9 + 1

(D) 117

(E) 1024

SOLN Note pattern in short staircases.

steps	ways	#
1	1	1
2	11, 2	2
3	111, 12, 21	3
4	1111, 112, 121, 211, 22	5
5	11111, 1112, 1121, 1211, 2111, 221, 212, 122	8

The pattern is the Fibonacci sequence; the 10th number is 89.

Alt. Soln: tally the combinations to climb 10 stairs and count ways each can happen.

$10 = 5 \cdot 2 + 0 \cdot 1 \to \binom{5}{0} = 1$

$10 = 4 \cdot 2 + 2 \cdot 1 \to \binom{6}{2} = 15$

$10 = 3 \cdot 2 + 4 \cdot 1 \to \binom{7}{4} = 35$

$10 = 2 \cdot 2 + 6 \cdot 1 \to \binom{8}{6} = 28$

$10 = 1 \cdot 2 + 8 \cdot 1 \to \binom{9}{8} = 9$

$10 = 0 \cdot 2 + 10 \cdot 1 \to \binom{10}{10} = 1$ □

37. Find the volume of the solid obtained by rotating the trapezoid around the y-axis.

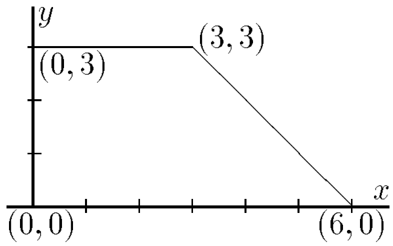
(A) $\frac{243}{4}\pi$

[(B)] 63π

(C) 72π

(D) 90π

(E) 108π



SOLN Geometry solution: the solid is a truncated cone. The volume of a cone is $\frac{1}{3}\pi r^2 h$. If extended, the full cone's apex would be at $(0, 6)$.

$$V_{\text{trunc}} = V_{\text{full}} - V_{\text{cut off}} =$$

$$\tfrac{1}{3}\pi(6)^2(6) - \tfrac{1}{3}\pi(3)^2(3) =$$

$$72\pi - 9\pi$$

Calculus way: integrate stacked disks:

$$V = \pi \int_0^3 (-y+6)^2 \, dy =$$

$$-\tfrac{\pi}{3}\left[(-y+6)^3\right]_0^3 = -9\pi + 72\pi$$

The integral can also be done with cylindrical shells. □

38. A basketball fieldhouse seats 15 000. Courtside seats sell for \$9, baseline for \$7, and balcony for \$4. The total revenue for a sell-out is \$81 000. If half the courtside and balcony seats and all the baseline seats are sold the total revenue is \$47 500. How many of each type of seat are there?

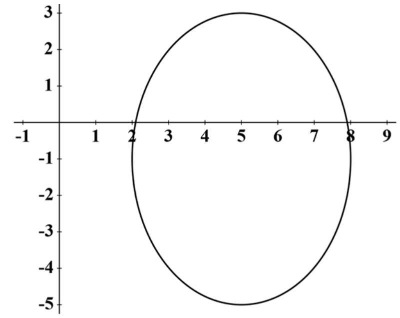
	courtside	baseline	balcony
(A)	4000	3000	8000
(B)	3200	1800	10 000
(C)	3000	3000	8000
[(D)]	3000	2000	10 000
(E)	3500	2500	9000

SOLN

$$x + y + z = 15\,000$$
$$9x + 7y + 4z = 81\,000$$
$$9(\tfrac{1}{2}x) + 7y + 4(\tfrac{1}{2}z) = 47\,500$$

$$x = 3000, \quad y = 2000, \quad z = 10\,000$$ □

39. Which equation best matches the graph?



(A) $\frac{(x-5)^2}{9} - \frac{(y+1)^2}{16} = 1$

(B) $\frac{(x-5)^2}{16} + \frac{(y+1)^2}{9} = 1$

(C) $\frac{(y+1)^2}{16} - \frac{(x+5)^2}{9} = 1$

(D) $\frac{(x+5)^2}{9} + \frac{(y-1)^2}{16} = 1$

[(E)] $\frac{(x-5)^2}{9} + \frac{(y+1)^2}{16} = 1$

SOLN $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$: ellipse with vertical axis, center $(h, k) = (5, -1)$. The major axis $2a$ is 8, so $a = 4$; the minor axis $2b$ is 6, so $b = 3$. □

40. How many 3-digit numbers that contain three different digits are between 300 and 800 and use only 1, 2, 3, 4, 5, 6, 7, 8, 9 ?

[(A)] 280

(B) 336

(C) 360

(D) 405

(E) 440

SOLN The first digit must be 3, 4, 5, 6, or 7. After one of these five numbers is chosen, only eight choices are left for the second digit, and seven choices for the third digit. $5 \times 8 \times 7 = 280$ □