Snow College Mathematics Contest

key

April 3, 2018 **Senior Division: Grades 10-12** **Form: T**

Bubble in the single best choice for each question you choose to answer.

1. If $\log_{10} 5 = 0.699$ what is $\log_{10} 500$?

   (A) 2.699 [boxed]

   (B) 5.699

   (C) 6.99

   (D) 69.9

   (E) 500.699

   SOLN $\log_{10} 500 = \log_{10}(5 \times 100) = \log_{10} 5 + \log_{10} 100 = 0.699 + 2$ □

2. Three dice are rolled. What is the probability that none of them show a 1 or 2?

   (A) $\frac{1}{27}$

   (B) $\frac{8}{27}$ [boxed]

   (C) $\frac{18}{27}$

   (D) $\frac{19}{27}$

   (E) $\frac{26}{27}$

   SOLN Probability of one die not showing 1 or 2 is $\frac{4}{6} = \frac{2}{3}$. Three dice: $\left(\frac{2}{3}\right)^3$. □

3. The number of spots on a particular normal Dalmatian dog is divisible by 3. When the number of spots is divided by the number of legs, a remainder of 3 results. The spots can also be divided by the total of legs, ears, eyes, and tail to leave a remainder of 6. What is the minimum number of spots?

   (A) 12

   (B) 15 [boxed]

   (C) 18

   (D) 27

   (E) 36

   SOLN The lcm of 3 and 4 is $2^2 \cdot 3^1 = 12$. Adding the remainder 3 gives 15. The lcm of 3 and 9 is $3^2 = 9$. Adding the remainder 6 gives 15 as before. The lcm of 3, 4, and 9 is $2^2 \cdot 3^2 = 36$ so other possible numbers are $15 + 36$ or, in general, $15 + 36n, \quad n = 0, 1, 2, 3, \ldots$. □

4. Define $3\mathbb{Z} = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$. If we define primes in $3\mathbb{Z}$ as those positive numbers that can<u>not</u> be expressed as products of smaller positive elements of the set, what is the sum of the first three positive *composite* (non-prime) numbers in $3\mathbb{Z}$?

   (A) 9

   (B) 18

   (C) 24

   (D) 36

   $\boxed{\text{(E)}}$ 54

   SOLN The composite numbers are 3×3, 3×6, 3×9, 3×12, 3×15, 3×18, ...

   $$3(3 + 6 + 9) = 3 \times 18 = 54$$

   The property of the uniqueness (up to ordering) of prime factorization does not hold in this ring; that is, the fundamental theorem of arithmetic does not apply to this system! For example, 3×12 and 6×6 are two distinct prime factorizations of 36 in $3\mathbb{Z}$. □

5. An equilateral triangle has a side of length a. What is the area of the largest circle which can be drawn within this triangle?

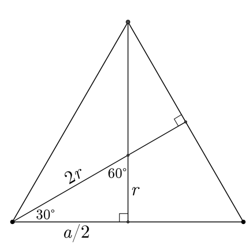
   (A) $\frac{\pi}{18}a^2$

   (B) $\sqrt{3}\pi a^2$

   (C) $2\sqrt{3}\pi a^2$

   $\boxed{\text{(D)}}$ $\frac{\pi}{12}a^2$

   (E) $\frac{3}{2}\sqrt{3}a^2$



   SOLN Draw a 30-60-90 right triangle inside with sides r and $\frac{a}{2}$ and hypotenuse $2r$.

   $\left(\frac{a}{2}\right)^2 = (2r)^2 - r^2 \Rightarrow r = \frac{a}{\sqrt{12}}$

   $A = \pi r^2 = \pi \left(\frac{a}{\sqrt{12}}\right)^2$

   Or use $\tan 30^\circ = 1/\sqrt{3} = r/(a/2)$. □

6. A sphere is inscribed in a cube. What is the ratio of the volume of the sphere to the volume of the cube?

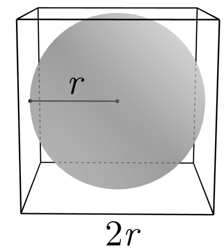
   $\boxed{\text{(A)}}$ $\frac{\pi}{6}$

   (B) $\frac{2\pi}{3}$

   (C) $\frac{\pi}{8}$

   (D) $\frac{6}{\pi}$

   (E) $\frac{\pi}{2}$



   SOLN Let r be the radius of the sphere. The sphere volume is $\frac{4}{3}\pi r^3$ and the cube volume will be $(2r)^3 = 8r^3$. The ratio of the sphere volume to the cube volume will be $\frac{\frac{4}{3}\pi r^3}{8r^3} = \frac{\pi}{6}$. □

7. Find the measure of an angle that is both the complement of $\angle A$ and the supplement of $\angle B$ if $m\angle A + m\angle B = 236^\circ$.

   $\boxed{\text{(A)}}$ 17°

   (B) 34°

   (C) 45°

   (D) 59°

   (E) 67.5°

   SOLN Let $\angle C$ be the angle in question. Then $m\angle A + m\angle C = 90^\circ$ and $m\angle B + m\angle C = 180^\circ$. $m\angle A + m\angle C + m\angle B + m\angle C = 270^\circ$. $2m\angle C = 270^\circ - 236^\circ$ □

8. Simplify.

   $$(\log_{624} 625)(\log_{623} 624) \cdots (\log_6 7)(\log_5 6)$$

   (A) 2

   (B) 2.5

   $\boxed{\text{(C)}}$ 4

   (D) 5

   (E) 6

   SOLN Change of base formula gives pattern where all will cancel except the 1st numerator and the last denominator leaving $(\log 625) / (\log 5) = \log_5 625$. □

9. A circle and a square have the same perimeter. Then

(A) their areas are equal.

[(B)] the area of the circle is the greater.

(C) the area of the square is the greater.

(D) the area of the circle is π times the area of the square.

(E) None of these

SOLN
$$A_\circ = \frac{P^2}{4\pi} \quad > \quad A_\square = \frac{P^2}{16}$$
For a given perimeter, a circle encloses the most area of any shape. □

10. How many two-digit whole numbers are exactly 7 times the sum of their digits?

(A) 0

(B) 1

(C) 2

(D) 3

[(E)] 4

SOLN $10a+b = 7(a+b) \implies 3a = 6b \implies a = 2b \qquad \{21, 42, 63, 84\}$ □

11. Convert the repeating decimal into a fraction. After reducing to lowest terms, find the difference between the denominator and the numerator. $0.60\overline{60}$

[(A)] 13

(B) 33

(C) 39

(D) 47

(E) 60

SOLN
$$\begin{aligned} 100x &= 60.\overline{60} \\ x &= 00.\overline{60} \\ \hline 99x &= 60 \end{aligned}$$
$\implies x = \frac{60}{99} = \frac{20}{33}$ □

12. Tau, who loves eating 2 pieces of pi, discovered that when the digits of a three-digit natural number are rearranged to form a second number, the difference between the two numbers is usually divisible by ____.

(A) 2

(B) 4

(C) 5

(D) 6

[(E)] 9

SOLN Original number: $abc = 100a+10b+c$
If $bac = 100b + 10a + c \quad \Delta = 90a - 90b$
If $bca = 100b + 10c + a$
$\Delta = 99a - 90b - 9c$
If $cab = 100c + 10a + b$
$\Delta = 90a + 9b - 99c$
If $cba = 100c + 10b + a \quad \Delta = 99a - 99c$
If $acb = 100a + 10c + b \quad \Delta = 9b - 9c$ □

13. Given three distinct one-digit numbers (i.e., from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$), what is the probability that two of them add up to the third one?

(A) $\frac{1}{10}$

[(B)] $\frac{4}{21}$

(C) $\frac{1}{3}$

(D) $\frac{2}{3}$

(E) $\frac{5}{27}$

SOLN There are ${}_9C_3 = 84$ possible trios. If 1 is the smallest of the three, the numbers $3, 4, 5, 6, 7, 8, 9$ are all possible sums, which gives 7 sums. If 2 is the smallest, there are 5 possible sums, etc. There are $7 + 5 + 3 + 1 = 16$ total possible sums, giving a probability of $\frac{16}{84} = \frac{4}{21}$. □

14. Given these rules, how many different paths can you take to spell SNOWMATH?

- Begin at the top
- Move only down
- For each move, go to one of the letters directly below

S_2
N_2 N_2
O_2 O_2 O_2
W_2 W_2 W_2 W_2
M_1 M_2 M_2 M_2 M_1
A_1 A_2 A_2 A_1
T_1 T_2 T_1
H H

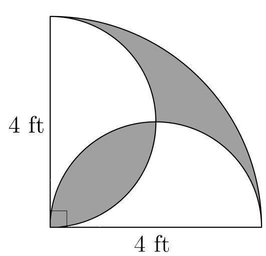
(A) 2
(B) 10
(C) 20
(D) 35
[(E)] 70

SOLN Each entry in Pascal's triangle is the number of ways to get to that spot in the triangle using the rules above. So there are 35 ways to get to each H: $2+(2+2)+(2+2+2)+(2+2+2+2)+(1+2+2+2)+(1+2+2)+(1+2) = 35$

1
1 **1**
1 **2** **1**
1 **3** **3** **1**
1 **4** **6** **4** **1**
1 **5** **10** **10** **5** 1
1 6 **15** **20** **15** 6 1
1 7 21 **35** **35** 21 7 1 □

15. Two semicircles are in a quarter-circle. What is the area of the shaded region in ft^2?

(A) $2\pi - 4$
[(B)] $4\pi - 8$
(C) $4\pi - 4$
(D) $2\pi + 4$
(E) $2\pi - 2$



SOLN Divide the overlap of the semicircles in half with a diagonal. Then rotate each half around the point of intersection creating a shaded region with a quarter-circle outer edge and a straight line below. The white triangle has an area of $8\,\text{ft}^2$ and the full quarter-circle has an area of $4\pi\,\text{ft}^2$. □

16. Towns A, B, and C are at the corners of a triangle with equal sides. A car travels at constant speed from A to B at 30 mph, from B to C at 40 mph, and from C back to A at 60 mph. What is the average speed for the round trip?

[(A)] 40 mph
(B) 43 mph
(C) 45 mph
(D) 48 mph
(E) 50 mph

SOLN The answer is the same for any size triangle, but let's assume a specific case for ease of illustration: $s = 120$ mi, so the total trip is 360 mi. The first leg takes $(120\,\text{mi})/(30\,\text{mi/h}) = 4\,\text{h}$; likewise, the second leg takes 3 h, and the last leg takes 2 h, for a total of 9 h. The average speed is $360\,\text{mi}/9\,\text{h} = 40\,\text{mi/h}$.

Marilyn vos Savant in Parade, Dec. 17, 2017. □

17. Matrix X is an inverse to matrix A if $AX = XA = I$. Which of the following can**not** have an inverse matrix?

(A) $\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$

(B) $\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$

[(C)] $\begin{bmatrix} 3 & -5 \\ -6 & 10 \end{bmatrix}$

(D) $\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

(E) $\begin{bmatrix} -1 & -2 \\ -3 & 4 \end{bmatrix}$

SOLN If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A) = ad - bc$. A is invertible iff $\det(A) \neq 0$. □

18. How many different positive integers less than 2018 are multiples of 20 or 18 (inclusive)?

(A) 198
[(B)] 201
(C) 207
(D) 212
(E) 252

SOLN The number of multiples of 20 less than 2018 is $\lfloor \frac{2018}{20} \rfloor = 100$. For 18, there are $\lfloor \frac{2018}{18} \rfloor = 112$. However, these sets overlap at the multiples of 180, of which there are $\lfloor \frac{2018}{180} \rfloor = 11$. $\therefore$ there are $100 + 112 - 11 = 201$ different numbers that are multiples of 20 or 18. □

19. What is $2018 \mod 26$?

(A) 77
[(B)] 16
(C) 18
(D) 0
(E) 1992

SOLN $2018 = 77 \times 26 + 16$ □

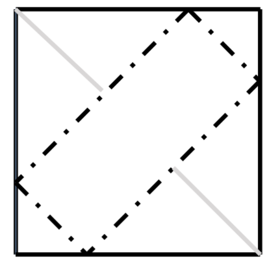
20. Driving on a certain street, Prof. B can hit stoplights green 90% of the time. What is the probability that he will hit exactly two of the next three lights green, assuming independent events?

(A) 0.81
(B) 0.081
[(C)] 0.243
(D) 0.600
(E) 0.900

SOLN Probability of hitting the first two lights green and the last not green is $0.90^2 \times 0.10$. There are $\binom{3}{1} = 3$ different (mutually exclusive) ways to do this. □

21. At a certain business, packages are delivered through a square delivery chute, with each side of the chute measuring 3 ft. Suppose a rectangular box of dimensions $6\,\text{ft} \times 4\,\text{ft} \times x$ is to be delivered through the chute. What is the maximum value of x?

(A) $\frac{1}{2}$
(B) $2\sqrt{3} - 3$
[(C)] $3\sqrt{2} - 4$
(D) $\sqrt{2} - 1$
(E) $2\sqrt{5} - 4$



SOLN If the long dimension of the inner rectangle is 4, then each of the gray lines must have a length of 2, as we are bisecting a 45-45-90 isosceles right triangle with hypotenuse 4. The diagonal of the chute is $\sqrt{18}$ or $3\sqrt{2}$, so the width of the box is $x = 3\sqrt{2} - 2(2)$. □

22. If $f(x) = 3x - 2$, find $f(f(f(3)))$.

(A) 19
[(B)] 55
(C) 75
(D) 107
(E) 163

SOLN $f(3) = 7 \quad f(7) = 19 \quad f(19) = 55$ □

23. Suppose $f(\ln x) = \sqrt{x}$. Find $f^{-1}(x)$.

[(A)] $\ln x^2$
(B) $(\ln x)^2$
(C) $e^{\sqrt{x}}$
(D) e^{x^2}
(E) $\sqrt{\ln x}$

SOLN If $f(\ln x) = \sqrt{x}$, then $y = f(x) = \sqrt{e^x}$. To find $f^{-1}(x)$, switch x and y and solve for y. To solve $x = \sqrt{e^y}$, we have $x^2 = e^y$ which gives $y = \ln x^2$. □

24. Simplify the sum $\sum_{n=1}^{115} i^n$ if $i = \sqrt{-1}$.

(A) -1 ☐
(B) 0
(C) i
(D) $1 - i$
(E) $-i$

SOLN $\sum_{n=1}^{4} i^n = i - 1 - i + 1 = 0$ and since $115 = 4(28) + 3$, $\sum_{n=1}^{115} i^n = i - 1 - i$. □

25. An aquarium on a level table has a square base 10 in wide and is 8 in tall. When tilted, the water in it just covers one of the 10 in × 8 in ends but only three-fourths of the square bottom. What is the depth of the water when the bottom is again level?

(A) 2 in
(B) 3 in ☐
(C) 3.25 in
(D) 4 in
(E) 6 in

SOLN When tilted, the water forms a triangular prism with volume $\frac{1}{2}(7.5\,\text{in})(10\,\text{in})(8\,\text{in}) = 300\,\text{in}^3$. With the square base $(10\,\text{in})(10\,\text{in}) = 100\,\text{in}^2$, the height will be 3 in. □

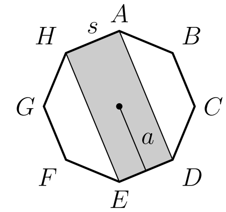
26. Suppose the Earth is a perfect sphere and that there is a steel belt fitting snugly around it at the equator. If the length of the belt were increased by 10 feet, how far above the Earth would the belt be raised if it remained circular and centered around the Earth?

(A) Less than 1 inch
(B) Between 1 inch and 2 inches
(C) Between 2 inches and 1 foot
(D) Between 1 foot and 2 feet ☐
(E) More than 2 feet

SOLN The change in the circumference of the belt is $\Delta C = 10$ and $\Delta C = 2\pi\Delta r$ so $\Delta r = \Delta C/2\pi$. □

27. In regular octagon $ABCDEFGH$, what is the fraction of the total octagon area found in rectangle $ADEH$?

(A) $\frac{1}{2}$ ☐
(B) $\frac{7}{12}$
(C) $\frac{3}{5}$
(D) $\frac{2}{3}$
(E) $\frac{1}{3}$



SOLN The area of a regular polygon is $\frac{1}{2}ap$ where a is the apothem length (perpendicular distance from center to one side) and p is the perimeter. If s is the length of a side, then $p = 8s$ and the area of the octagon is $4as$. The rectangle width will be s and length $2a$. □

28. Shannon takes her favorite number, adds 5 to it, multiplies the answer by 10, subtracts 20 from the result and then drops the final 0. If Shannon's (correct) answer is 9, what is her favorite number?

(A) 5
(B) 6 ☐
(C) 7
(D) 8
(E) 9

SOLN $9 = \frac{10(x+5)-20}{10} = x + 3 \implies x = 6$ □

29. If the standard order of operations is reversed (addition and subtraction are done first and exponentiation is done last), what is the value of $2 \cdot 3\hat{}2 + 3$?

(A) 21
(B) 24
(C) 39
(D) 486
(E) 7776 ☐

SOLN $(2 \times 3)\hat{}(2 + 3) = 6^5 = 7776$ □

30. Simplify. $\dfrac{\tan t - \sin t \cos t}{\tan t}$

(A) $\sin t$

[(B)] $\sin^2 t$

(C) $\cos t$

(D) $\cos^2 t$

(E) 1

SOLN Use $\tan t = \sin t / \cos t$ and simplify the fraction. $1 - \cos^2 t = \sin^2 t$. □

31. Different shades of pink, red, and white can be made by mixing whole numbers of quarts of red and white paint. Shades are different if the ratio of red to white is different. Find the number of different possible shades that can be made from at most 4 quarts of red and 5 quarts of white paint.

(A) 15

(B) 16

[(C)] 17

(D) 18

(E) 19

red	white
0	any
1	0, 1, 2, 3, 4, 5
2	1, 3, 5
3	1, 2, 4, 5
4	1, 3, 5

SOLN $1 + 6 + 3 + 4 + 3 = 17$ □

32. How many strings of length 7 of a's, b's, and c's are there such that all the a's, if any, show up at the beginning? For example: *aaaabcb* is acceptable but *abbbbab* is not.

(A) 254

[(B)] 255

(C) 256

(D) 257

(E) 258

SOLN The sum of a finite geometric sequence $1 + 2^1 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 = 2^8 - 1 = 255$. □

33. Eric the Sheep is at the end of a line of 50 sheep waiting to be shorn. But being impatient, Eric sneaks up the line two places every time the shearer takes a sheep from the front to be shorn. So, for example, while the first sheep is being shorn, Eric moves ahead so that there are two sheep behind him in line. If at some point it is only possible for Eric to move one place, he does so. How many sheep get shorn **before** Eric?

(A) 15

(B) 16

[(C)] 17

(D) 18

(E) 19

SOLN When the 1st sheep is taken, Eric moves to number 48. When the 16th sheep is taken Eric has moved up 32 places, making him #18. There is one sheep (#17) ahead of him. When the shearer takes number 17, Eric is next. □

34. Find the values of c such that the trinomial $x^2 + cx - 14$ can be factored over integers.

(A) $\{-15, -9, 9, 15\}$

(B) $\{-10, -4, 5, -19\}$

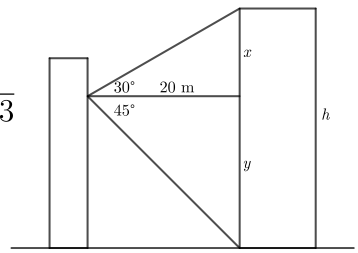
(C) $\{-14, -13, 13, 14\}$

[(D)] $\{-13, -5, 5, 13\}$

(E) $\{-15, -13, -9, -5, 5, 9, 13, 15\}$

SOLN The possible combinations of factors of -14 are: -1 and 14, 1 and -14, -2 and 7, and 2 and -7. The sums of the pairs are the possible values for c. □

35. From my balcony, I measure the angle of elevation to the top of the neighboring building to be 30°, and the angle of depression to the base to be 45°. If my (perpendicular) distance from the building is 20 m, how tall is the building in meters?



- **(A)** $20 + \frac{20\sqrt{3}}{3}$
- (B) $20 + \sqrt{3}$
- (C) $20 + 20\sqrt{3}$
- (D) $\frac{40}{3} + \frac{20\sqrt{3}}{3}$
- (E) $\frac{40}{3} + \frac{40\sqrt{3}}{3}$

SOLN The elevation of 30° makes a 30-60-90 triangle and the depression of 45° makes a 45-45-90 triangle. The opposite sides, x and y in the figure, make up the height of the building. Since the legs of the 45-45-90 triangle are the same length, $y = 20$ m. In a 30-60-90 triangle the sides are x, $2x$, and $\sqrt{3}x$, so $\sqrt{3}x = 20$ m, so $x = \frac{20}{\sqrt{3}} = \frac{20\sqrt{3}}{3}$. □

36. Simplify completely:

$$\sqrt[3]{\sqrt[5]{\sqrt{x^{40}}}}$$

- (A) $\sqrt[30]{x^{40}}$
- (B) $\sqrt[15]{x^{20}}$
- (C) $x^{10}\sqrt[40]{x^{10}}$
- (D) $x\sqrt[30]{x}$
- **(E)** $x\sqrt[3]{x}$

SOLN $(((x^{40})^{1/2})^{1/5})^{1/3} = x^{40/30} = \sqrt[3]{x^4}$ □

37. If 10, $2x$, and $3x$ are the first three terms of a geometric sequence, what is x?

- (A) $\frac{3}{2}$
- (B) 5
- (C) $\frac{2}{3}$
- **(D)** $\frac{15}{2}$
- (E) $\frac{5}{3}$

SOLN The ratio between any two successive elements must be the same.

$$\frac{2x}{10} = \frac{3x}{2x} \implies x = \left(\frac{3}{2}\right)\left(\frac{10}{2}\right)$$

□

38. Car A is 2 mi ahead of car B, which is going in the same direction. 8 min later car A is only 1 mi ahead of car B. On average, how much faster is car B traveling?

- (A) 5 mph
- **(B)** 7.5 mph
- (C) 10 mph
- (D) 15 mph
- (E) not enough info

SOLN Let d_i and v_i be the distance and speed car$_i$ travels in the 8 min. Then $d_A = 8v_A$ and $d_B = 8v_B = d_A + 1$. Substitute and subtract to get $8(v_B - v_A) = 1 \Rightarrow (v_B - v_A) = \frac{1}{8}\frac{\text{mi}}{\text{min}} = \frac{60}{8}$ mph.
Shorter: Car B travels 1 mi more in 8 min: $\frac{1}{8}\frac{\text{mi}}{\text{min}} = \frac{60}{8}$ mph $= 7.5$ mph □

39. A roll of paper towels contains 100 paper towels. The tube around which the towels are rolled is 2 cm in diameter. Including the tube, the whole roll is 14 cm in diameter. What is the diameter of the roll (and tube) when there only 50 paper towels left?

(A) 6 cm

(B) 7 cm

(C) 8 cm

[(D)] 10 cm

(E) 12 cm

SOLN The annular cross-sectional area of the paper towels on the full roll is $(7^2 - 1^2)\pi\,\text{cm}^2 = 48\pi\,\text{cm}^2$. If the new radius of the roll is R, then $(R^2 - 1^2)\pi\,\text{cm}^2 = 24\pi\,\text{cm}^2 \implies R = 5\,\text{cm}$ □

40. During shooting practice, a basketball player takes one step closer if she misses a shot, and one step farther away if she makes a shot. After a while, she notices she is two steps farther away than when she began. What is the most we can say about her shooting percentage P (i.e., shots made ÷ shots taken)?

(A) $25\% < P \leq 50\%$

[(B)] $P > 50\%$

(C) $P > 67\%$

(D) $P > 75\%$

(E) not enough info

SOLN If she's farther away, then $P > 50\%$. Without knowing how many shots she took, however, there's no way to know how much better than 50%. If she took just two shots $P = 100\%$; if she took thousands, P is slightly above 50%. □