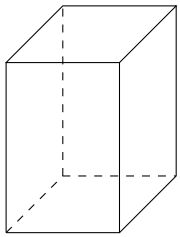
Snow College Mathematics Contest

March 19, 2024 **Senior Division: Grades 10-12** **Form: T**

Bubble in clearly the single best choice for each question you choose to answer.

1. For a science project, you decide to create a sky lantern. A wire frame is built with a square base as shown below. Before you open it, you measure the width of the garbage bag that you will use to be 40 in while still closed. You open the bag and pull it over the frame so the opening of the bag fits snugly at the square base. What is the length of the diagonals of the square base?



(A) 28.3 in
(B) 19.3 in
(C) 26.9 in
(D) 30.4 in
(E) 54.4 in

2. Jane works at a car wash. 7 cars pull up at the same time. 3 are red, 2 are green, and 2 are blue. Her car wash has 2 bays. How many different ways can 2 cars that are the same color be selected to be washed at the same time — one in each bay?

(A) 3
(B) 4
(C) 5
(D) 6
(E) 7

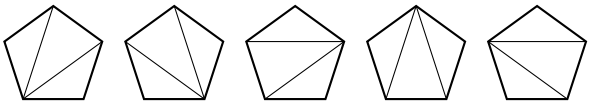
3. If you flip a coin 3 times and then repeat the experiment, what is the probability of getting exactly 2 heads out of 3 both times?

(A) $\frac{2}{3}$
(B) $\frac{15}{64}$
(C) $\frac{6}{64}$
(D) $\frac{9}{64}$
(E) $\frac{24}{64}$

4. Dunddin Dede does not dislike Dumdums. A letter is drawn from the previous sentence. If it is a vowel, we return the letter and draw again. What is the probability that we end up with the chosen letter being d or D?

(A) $\frac{9}{20}$
(B) $\frac{9}{31}$
(C) $\frac{1}{12}$
(D) $\frac{1}{8}$
(E) $\frac{8}{31}$

5. A polygon with $n + 2$ sides can be cut into n triangles. The number of different ways to cut the polygon into triangles using line segments from one vertex to another generates a sequence called the *Catalan numbers*: 1, 2, 5, ... For example, a square can be cut into two triangles in two different ways. A pentagon can be cut into 3 triangles in five different ways as illustrated.



In how many different ways can a hexagon be cut into 4 triangles?

(A) 7
(B) 11
(C) 14
(D) 42
(E) 55

6. If $\log_2(\log_3(\log_4 x)) = \log_3(\log_4(\log_2 y)) = \log_4(\log_2(\log_3 z)) = 0$, then what is the value of $x + y + z$?

(A) 50

(B) 58

(C) 89

(D) 111

(E) 1296

7. An ambitious chef wants to make circular pizzas on her square baking sheet. One large pizza is going to fill as much space as possible. Then a tiny pizza, as large as possible, will be placed in each of the four corners. The baking sheet has sides of 2 ft. What is the area of one of the tiny pizzas in sq. ft?

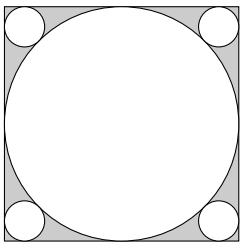
(A) $\pi - \sqrt{2}$

(B) $\pi(\sqrt{2} - 1)^4$

(C) $\pi(\sqrt{2} - 1)^2$

(D) $4\pi(2 - \sqrt{2})$

(E) $2\pi(2 - \sqrt{2})$



8. One side of the gray square is increased by 3 cm while its adjacent side is decreased by 2 cm. The perimeter of the resulting rectangle is 22 cm. What is the area of the original gray square?

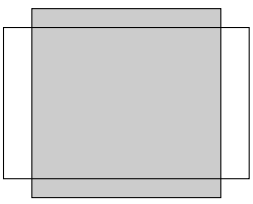
(A) $9\,\text{cm}^2$

(B) $16\,\text{cm}^2$

(C) $25\,\text{cm}^2$

(D) $64\,\text{cm}^2$

(E) $121\,\text{cm}^2$



9. Let a_n be the remainder of p^n divided by q. Note that a_n will be a repeating pattern for all integer values of p and q. What is the remainder of $(2^{241} - 2)$ divided by 5?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

10. Find the value of the shaded area assuming that the triangles are equilateral and the pattern continues indefinitely.

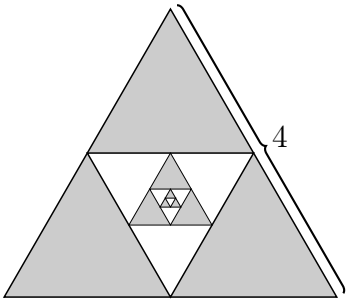
(A) $\frac{4\sqrt{3}}{3}$

(B) $\frac{7\sqrt{3}}{2}$

(C) $\frac{5\sqrt{3}}{4}$

(D) $\frac{16\sqrt{3}}{3}$

(E) $\frac{16\sqrt{3}}{5}$



11. A basketball player makes free throws 80% of the time. He plans to shoot 5 free throws. What is the probability that he will make exactly 4 of the 5 attempts?

(A) 41%

(B) 50%

(C) 80%

(D) 62%

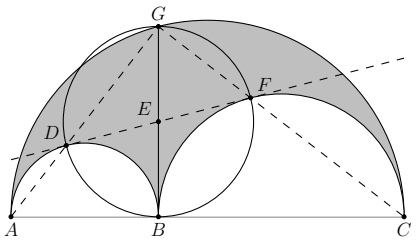
(E) 33%

12. In geometry, an *arbelos* is a plane region bounded by three semicircles such that each corner of each semicircle is shared with one of the others, all on the same side of a straight line that contains their diameters. This is the shaded region illustrated below.
Some remarkable properties of the arbelos are that

- $BFGD$ is a rectangle
- line DF is tangent to the smaller semicircles
- the area of the arbelos equals the area of the circle with diameter BG.

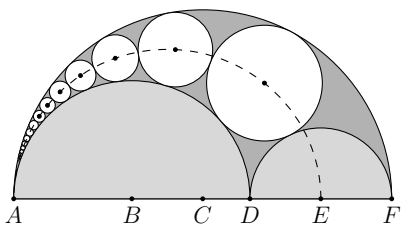
If $AB = 2$ and $AC = 8$, what is the area of the arbelos in square units?

(A) π
(B) 2π
(C) 3π
(D) 5π
(E) 10π



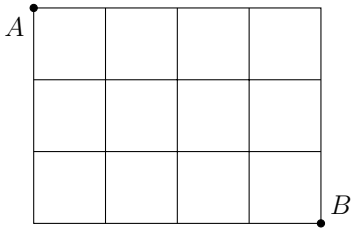
13. A *Pappus chain* is a sequence of circles tangent to the generating circles of an arbelos (dark shading) as illustrated below. The points B, C, and E are the centers of the circles generating the arbelos. The centers of the circles in the chain lie along an ellipse (dashed) with focal points B and C. If $AD = 6$ and $AF = 8$, what is the semimajor axis length of the ellipse?

(A) 2.5
(B) 3
(C) 3.5
(D) 4
(E) 4.5



14. Going only right or down, how many different ways are there to get from point A (upper left corner) to point B (lower right corner) of the 3×4 grid below?

(A) 28
(B) 32
(C) 35
(D) 56
(E) 84



15. How many pairs of integers a and b are solutions to the equation?

$$a(a+1)(a+2) = b^2 + 4$$

Hint: use divisibility rules.

(A) 0
(B) 1
(C) 2
(D) 3
(E) ∞

16. One extension of the real numbers is the complex numbers $a + b\mathrm{i}$, where a, b are real numbers and $\mathrm{i}^2 = -1$. Another is the dual numbers $a + b\varepsilon$, where a, b are real numbers and $\varepsilon^2 = 0$. Use the same technique you use to divide complex numbers to compute $(4 + 3\varepsilon) \div (2 + 5\varepsilon)$.

(A) $\frac{5}{2} - \frac{3}{4}\varepsilon$
(B) $3 + 2\varepsilon$
(C) $\frac{5}{3} - 4\varepsilon$
(D) $\frac{4}{5} + 3\varepsilon$
(E) $2 - \frac{7}{2}\varepsilon$

17. Simplify.

$$\sqrt{\mathrm{i}\sqrt{\mathrm{i}\sqrt{\mathrm{i}\ldots}}}$$

(A) $\sqrt{2\mathrm{i}}$

(B) $\mathrm{i}\sqrt{\mathrm{i}}+\mathrm{i}$

(C) $-\sqrt{2\mathrm{i}}$

(D) $-2\mathrm{i}$

(E) i

18. If $y=u^3+u$ and $u=x^2-1$, find $\frac{dy}{dx}$.

(A) $6x^5+2x$

(B) x^3-2

(C) $3x^5+12x^3$

(D) $8x^5+12x$

(E) $6x^5-12x^3+8x$

19. Compute the limit (if it exists).

$$\lim_{x\to\infty}(\sqrt{x^2+x+1}-\sqrt{x^2-x})$$

(A) 0

(B) 1/2

(C) 1

(D) 3/2

(E) The limit does not exist.

20. An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y=27-x^2$. Find the largest area the triangle can have.

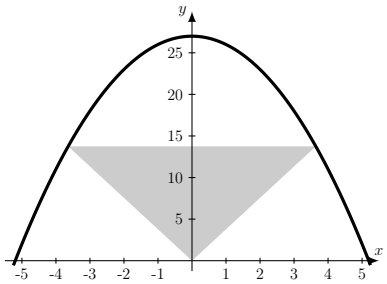
(A) $20\sqrt{3}$

(B) $\frac{45\sqrt{3}}{4}$

(C) 48

(D) 54

(E) 60



21. What is the area of the triangle?

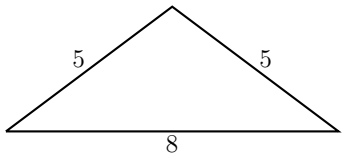
(A) 12

(B) 12.5

(C) 20

(D) 25

(E) 40



22. What is the sum of all odd numbers from 1 to 1999? (Hint: $1+3=4$, $1+3+5=9$, $1+3+5+7=16$.)

(A) 999 500

(B) 1 000 000

(C) 1 999 000

(D) 3 996 001

(E) 4 000 000

23. How many solutions does the system of equations have?

$$\begin{cases} x+3y=0 \\ 2x-3y=9 \end{cases}$$

(A) 0

(B) 1

(C) 2

(D) 3

(E) infinitely many

24. How many solutions does the system of equations have?

$$\begin{cases} x^2+y^2=9 \\ -x^2+(y-5)^2=4 \end{cases}$$

(A) 0

(B) 1

(C) 2

(D) 3

(E) infinitely many

25. What is the sum of the roots of the following polynomial?

$$x^6-8x^5-51x^4+302x^3+260x^2-1944x+1440$$

(A) -1994

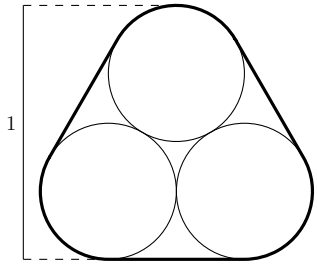
(B) -12

(C) -8

(D) 4

(E) 8

26. Three disks of equal radius are mutually tangent as in the figure below. A rubber band is wrapped around the outside of the group. The distance from the base to the top is exactly 1. Find the total length of the band.



(A) $2\pi + \sqrt{3}$

(B) 3π

(C) $\frac{3+2\pi}{\pi-\sqrt{3}}$

(D) $\frac{6+2\pi}{2+\sqrt{3}}$

(E) 6π

27. A random sample of 100 students polled about pizzas they like gave these results:

- 11 picked only ham & pineapple
- 32 picked only pepperoni
- 23 picked only supreme
- 8 picked ham & pineapple as well as pepperoni
- 10 picked pepperoni as well as supreme
- 7 picked ham & pineapple as well as supreme
- 9 picked all three types of pizza

What is the probability that a randomly chosen student likes ham & pineapple?

(A) 0.35

(B) 0.11

(C) 0.07

(D) 0.09

(E) 0.54

28. Three friends (Larry, Moe, Curly) went to Shemp's Grill together. They each ordered a different entreé (cheeseburger, chicken nuggets, or a salad) and a different flavor (vanilla, chocolate, or strawberry) of ice cream. The clues below help determine who ordered which entreé and ice cream flavor.

- The three friends are Curly, the one who ordered a cheeseburger, and the one who ordered chocolate ice cream.
- Moe did not order a salad.
- The chicken nuggets were ordered by either Larry or the one who ordered the chocolate ice cream (but not both).
- The one who ordered a salad does not like vanilla ice cream.

Which of the statements is true?

(A) Moe ordered vanilla ice cream.

(B) Curly ordered chicken nuggets.

(C) Curly ordered vanilla ice cream.

(D) Larry ordered a cheeseburger.

(E) Larry ordered strawberry ice cream.

29. Jen, Connor, and Bob are each collecting lapel pins. Bob has twice as many pins as Jen. Connor has three fewer pins than Bob. Jen's pins are eight more than one-third of Connor's pins. What is the total number of pins in the three collections combined?

(A) 95

(B) 39

(C) 63

(D) 81

(E) 102

30. Integer n is a solution to $8^n+4^n+2^n+1 = 585$. What is the value of $\sin\left(\frac{n\pi}{3}\right)$?

(A) -1

(B) $-\frac{\sqrt{2}}{2}$

(C) $-\frac{1}{2}$

(D) 0

(E) $\frac{\sqrt{3}}{2}$

31. Suppose that a dataset has a mean of x and a standard deviation of y. We transform the dataset by multiplying each value by 3 and adding 4. Ex: 5 would be transformed to $3 \cdot 5 + 4 = 19$. What are the mean and standard deviation of the new dataset?

(A) x, y

(B) $3x, 4y$

(C) $4x, 3y$

(D) $(3x + 4), (3y + 4)$

(E) $(3x + 4), 3y$

32.
$$\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right) =$$

(A) 0

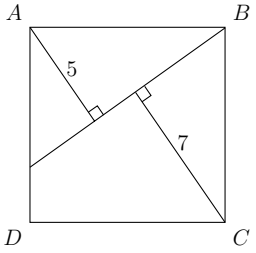
(B) $\frac{\pi}{4}$

(C) $-\frac{\pi}{4}$

(D) $\frac{3\pi}{4}$

(E) 1

33. Let $ABCD$ be a square, and l be a line segment from B to a point on side AD. If A is 5 cm from l and C is 7 cm from l, find the area of $ABCD$.



(A) 74

(B) 76

(C) 72

(D) 81

(E) 64

34. What is the binary representation of the base ten number 2024?

(A) 11111101000

(B) 111111001100

(C) 11110110000

(D) 11111010100

(E) 11110110100

35. A *proper factor* of a positive integer is any factor of that number, excluding the number itself. For example, the proper factors of 9 are 1 and 3. An *abundant number* is a whole number for which the sum of its proper factors is greater than the number itself. What is the sum of the two smallest abundant numbers?

(A) 24

(B) 30

(C) 32

(D) 38

(E) 40

36. A *Collatz sequence* starts with a positive integer n. If n is odd, then the next number in the sequence is $3n + 1$, but if n is even then perform $n/2$. Repeat the rule with each result. An as-yet-unproven, but so-far-un-excepted, conjecture is that all such sequences eventually arrive at 1. The number of steps required is not easily predictable. For example the sequence starting with 9 is 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. The assumption that larger numbers have longer sequences is not always true. Which of the numbers below has a longer sequence than for $n = 9$?

(A) 11

(B) 13

(C) 17

(D) 19

(E) 21

37. Given three positive integers, x, y, z, consider the question: Is $x - y$ odd?
Possibly useful information:

- Statement 1: $x = z^2$
- Statement 2: $y = (z-1)^2$

What is true about our ability to answer the question using statements 1 and/or 2?

(A) Only the information in statement 1, taken alone, is sufficient.

(B) Only the information in statement 2, taken alone, is sufficient.

(C) The information in either statement 1 or 2, taken alone, is sufficient.

(D) The information in statements 1 & 2, taken together, is sufficient.

(E) Even taking both statements together, there is insufficient information to answer the question.

38. Compute the radius of the circle with equation $x^2 + 12x + y^2 + 4y = -4$.

(A) 5.5

(B) 6

(C) 7

(D) $7\sqrt{2}$

(E) 8

39. For the function $f(x) = x^2+2x-5$, compute the value of $f(f(f(1)))$.

(A) -5

(B) 5

(C) 10

(D) 12

(E) 115

40. Find the intersection point of the diagonals of the parallelogram $ABCD$ for $A(2,-1)$, $B(5,2)$, $C(7,-3)$, and $D(4,-6)$.

(A) $(\frac{9}{2}, -\frac{5}{2})$

(B) $(4,-2)$

(C) $(5,-3)$

(D) $(\frac{9}{2}, -3)$

(E) $(\frac{9}{2}, -2)$