

# Snow College Mathematics Contest

key

March 24, 2026

Senior Division: Grades 10-12

Form: T

Bubble in clearly the single best choice for each question you choose to answer.

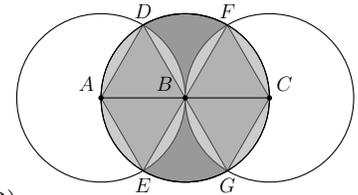
1. Suppose  $\mathbf{u} = u_1\hat{i} + u_2\hat{j}$  and  $\mathbf{v} = v_1\hat{i} + v_2\hat{j}$  are non-zero vectors and their dot product is  $\mathbf{u} \cdot \mathbf{v} = 0$ . Which must be true?

- (A)  $\frac{u_2}{u_1} \cdot \frac{v_2}{v_1} = -1$   
 (B)  $\frac{u_2}{u_1} \cdot \frac{v_2}{v_1} = 0$   
 (C)  $u_1v_2 = 0$  and  $u_2v_1 = 0$   
 (D)  $u_1v_2 = -v_1u_2$   
 (E)  $\|\mathbf{u}\|\|\mathbf{v}\| \leq 0$  where  $\|\mathbf{u}\|$  is the magnitude of  $\mathbf{u}$ .

SOLN The condition  $\vec{u} \cdot \vec{v} = 0$  implies  $u_1v_1 + u_2v_2 = 0$ . Thus, their slopes are negative reciprocals of each other, i.e.,  $\frac{u_2}{u_1} = -\frac{v_1}{v_2}$ . □

2. Points  $A$ ,  $B$ , and  $C$  are collinear and each is the center of a circle of radius 1. The outer circles pass through point  $B$  and the middle circle passes through points  $A$  and  $C$ . What is the total shaded area?

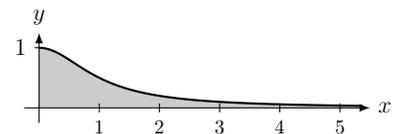
- (A) 3  
 (B)  $\sqrt{3}$   
 (C)  $\frac{\pi}{3}$   
 (D)  $\pi$   
 (E)  $(\sqrt{3} - \pi/3)$



SOLN First connect each circle's center to other centers, intersection points, points of tangency, etc. Then break down areas into circular segments and triangles. Each triangle is equilateral with side length 1 and area  $\frac{\sqrt{3}}{4}$ . The segment area can be found as a polar integral or by the formula  $\frac{1}{2}r^2(\theta - \sin\theta)$ . The angle is  $\pi/3$ . Subtract the area of four triangles and eight segments from the area of the full circle. Alternatively: A segment area can be found by subtracting the area of an equilateral triangle from  $1/6$  the area of a circle. □

3. Find the area between the function  $y = \frac{1}{x^2+1}$  and the  $x$ -axis from 0 to  $\infty$ .

- (A) 1  
 (B)  $\pi/2$   
 (C)  $\infty$   
 (D)  $\sqrt{2}$   
 (E)  $\pi\sqrt{2}/2$



SOLN Improper integral, so

$$\lim_{b \rightarrow \infty} \int_0^b \frac{1}{x^2+1} dx = \lim_{b \rightarrow \infty} [\tan^{-1} x]_0^b$$

□

4. One extension of the real numbers is the complex numbers  $a + bi$ , where  $a, b$  are real numbers and  $i^2 = -1$ . Another is the dual numbers  $a + b\varepsilon$ , where  $a, b$  are real numbers and  $\varepsilon^2 = 0$ . Which statement about  $f(a+b\varepsilon)$  is true? (Hints: Check a simple case such as  $f(x) = x^2$ .  $f'$  means derivative.)

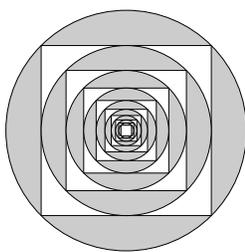
- (A)  $f(a + b\varepsilon) = f(a) + bf'(a)\varepsilon$   
 (B)  $f(a + b\varepsilon) = f(b) + af'(a)\varepsilon$   
 (C)  $f(a + b\varepsilon) = bf'(a)\varepsilon$   
 (D)  $f(a + b\varepsilon) = af'(b)\varepsilon$   
 (E)  $f(a + b\varepsilon) = af'(b)\varepsilon + bf'(a)\varepsilon$

**SOLN** This is called autodifferentiation.

For  $f(x) = x^2$ ,  $f(a + b\varepsilon) = (a + b\varepsilon)(a + b\varepsilon) = a^2 + 2ab\varepsilon$ . (N.B.  $f$  must be analytic.)  $\square$

5. A square is inscribed in a unit circle centered at the origin. Inscribed in the square is a second circle also centered at the origin. Find the sum of the gray areas if this process is repeated indefinitely.

- (A)  $2(\pi - 2)$   
 (B)  $\pi - \sqrt{2}$   
 (C)  $\pi - 2$   
 (D)  $2(\pi - \sqrt{2})$   
 (E)  $\pi/2 + 2\sqrt{2}/3$



**SOLN** The area of the circle of radius  $r$  minus the area of the inscribed square is  $\pi r^2 - (2r/\sqrt{2})^2 = r^2(\pi - 2)$ . The circles inscribed in the squares have radius  $r/\sqrt{2}$ . The total gray area is  $(\pi - 2)(1^2 + (1/\sqrt{2})^2 + (1/2)^2 + (1/2\sqrt{2})^2 + \dots)$  which is a geometric series  $(\pi - 2)(1 + 1/2 + 1/4 + 1/8 + \dots) = \frac{(\pi - 2)}{1 - 1/2} = 2(\pi - 2)$ .  $\square$

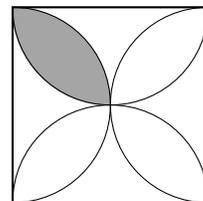
6. How many different 5-card hands can be drawn from a standard 52-card deck?

- (A)  $\frac{47!}{5! 52!}$   
 (B)  $\frac{47!}{5!}$   
 (C)  $\frac{52!}{47!}$   
 (D)  $\frac{52!}{5! 47!}$   
 (E)  $\frac{52!}{5!}$

**SOLN** This is a combination problem, 52 choose 5 is  $\frac{52!}{5! 47!}$ .  $\square$

7. What is the area of the shaded region if the inside curves are semicircles inscribed in a square of side length 2?

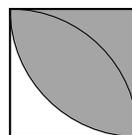
- (A)  $2 - \frac{\pi}{2}$   
 (B)  $1 - \frac{\pi}{4}$   
 (C)  $\pi - \frac{1}{2}$   
 (D)  $\frac{\pi}{2} - \frac{1}{2}$   
 (E)  $\frac{\pi}{2} - 1$



**SOLN** Since the side-length of the square is 2, the radii of the semicircles are each equal to 1. This means the area of the full circle is  $\pi$ , so a quarter of one of these circles has an area of  $\pi/4$ . Look at the top-left quadrant of the original diagram.



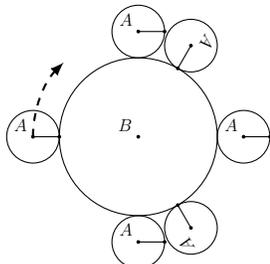
This shows that the white region in each of the two diagrams is  $1 - \pi/4$ .



So the shaded area of the original diagram is  $1 - 2(1 - \pi/4) = \pi/2 - 1$ .  $\square$

8. The radius of circle A is  $\frac{1}{3}$  the radius of circle B. Starting from the position shown, circle A rolls without slipping around circle B. When circle A returns to its original position how many rotations will it have made?

- (A) 4  
 (B)  $4\frac{1}{3}$   
 (C)  $4\frac{2}{3}$   
 (D) 5  
 (E) 6



**SOLN** Imagine a radius of circle A initially touching the circle B. As circle A is moved around circle B that line is again touching circle B after circle A has gone  $\frac{1}{3}$  of the way around. But the circle goes only  $\frac{1}{4}$ th the way around before completing one full revolution. <https://www.youtube.com/watch?v=FUHkTs-Ipfg> □

9. Each Fibonacci number  $F_n$  is the sum of the two previous ones:  $F_n = F_{n-1} + F_{n-2}$ , with  $F_1 = 1$ ,  $F_2 = 1$ . Then,  $F_3 = 2$ ,  $F_4 = 3$ ,  $F_5 = 5, \dots$ . Use the remarkable fact that

$$\gcd(F_m, F_n) = F_{\gcd(m,n)}$$

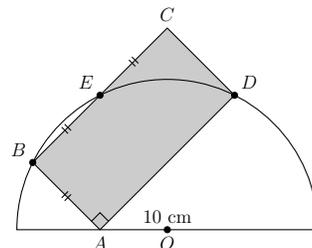
(where gcd is the greatest common divisor) to find the gcd of 144 and 610.

- (A) 2  
 (B) 4  
 (C) 6  
 (D) 8  
 (E) 12

**SOLN** This remarkable fact is Arthur Benjamin's **F**avorite **F**ibonacci **F**act.  $F_{12} = 144$  and  $F_{15} = 610$ , so  $\gcd(144, 610) = F_{\gcd(12,15)} = F_3 = 2$  (This can also be simply solved with the Euclidean algorithm.) □

10. Rectangle  $ABCD$  intersects a circle with diameter of 10 cm at the points  $B, E$ , and  $D$ . Segments  $AB, BE$ , and  $EC$  are congruent. What is the area of the rectangle?

- (A)  $10 \text{ cm}^2$   
 (B)  $20 \text{ cm}^2$   
 (C)  $22 \text{ cm}^2$   
 (D)  $2\sqrt{10} \text{ cm}^2$   
 (E)  $4\sqrt{10} \text{ cm}^2$



**SOLN**  $CDE$  is a  $45^\circ - 45^\circ - 90^\circ$  triangle so  $m\angle BED = 135^\circ$ .  $\angle BED$  is an inscribed angle of the circle so it subtends an arc of twice its size or  $270^\circ$ . Thus arc  $BED$  measures  $90^\circ$ . Segments  $OB$  and  $OD$  are both radii with length 5 cm and central angle  $BOD$  is a right angle. The rectangle's diagonal  $BD$  then measures  $5\sqrt{2}$ . Let the length of  $AB$  be  $x$  and  $BC$  be  $2x$  and  $AC$  is another diagonal with length  $5\sqrt{2}$ . Using the Pythagorean theorem, we find that  $x = \sqrt{10}$  and the area of the rectangle is  $\sqrt{10} \cdot 2\sqrt{10} = 20$ .

11. Solve the following equation for  $x$ .

$$\sqrt{\sqrt{\frac{x}{\sqrt{\frac{x}{\sqrt{x}}}}}} = 32$$

- (A)  $2^4$   
 (B)  $2^8$   
 (C)  $2^{10}$   
 (D)  $2^{12}$   
 (E)  $2^{16}$

**SOLN** Rewrite 32 as  $2^5$ . Square both sides. Do it again, again, and again.

$$\sqrt{\frac{x}{\sqrt{\frac{x}{\sqrt{x}}}}} = 2^{10}$$

$$\frac{x^2}{\frac{x}{\sqrt{\frac{x}{\sqrt{x}}}}} = 2^{20}$$

$$\frac{x^4}{\frac{x^2}{\frac{x}{\sqrt{x}}}} = 2^{40}$$

$$\frac{x^8}{\frac{x^4}{x}} = 2^{80}$$

The left side collapses down to  $x^5$  and the solution is  $x = (2^{80})^{1/5} = 2^{16}$ .  $\square$

12. Solve the equation.  $|x| = -x$

- (A)  $x = 0$   
 (B)  $(-\infty, 0]$   
 (C)  $(-\infty, 0)$   
 (D)  $(0, \infty)$   
 (E)  $[0, \infty)$

**SOLN** The absolute value of  $x$  is  $x$  when  $x \geq 0$  and is  $-x$  when  $x < 0$ . Thus  $x = 0$  and any value  $x < 0$  will satisfy this inequality.  $\square$

13. Solve the equation for  $x$ .

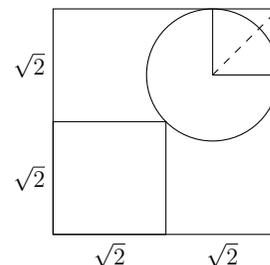
$$\frac{9^{x-5}}{4^{x-8}} = 144$$

- (A)  $\log_4 9$   
 (B)  $144 \log_9 4$   
 (C) 6  
 (D) 8  
 (E) 9

**SOLN** The equation can be solved with logarithms, but easier is that the powers of prime factors must be the same on both sides.  $144 = 2^4 \cdot 3^2$ , so the power of 3 on the left side must also be 2.  $9^{x-5} = 3^{2(x-5)} \implies 2(x-5) = 2$ . Solving using powers of 2 on both sides must also yield the same value for  $x$ .  $\square$

14. Find the diameter of the circle in the diagram.

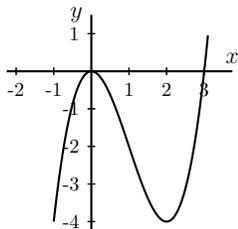
- (A)  $2/(1 + \sqrt{2})$   
 (B)  $4/(1 + \sqrt{2})$   
 (C)  $2\sqrt{2}$   
 (D)  $2\sqrt{2}/(1 + \sqrt{2})$   
 (E)  $3/2$



**SOLN** The diagonal of the small square is 2 and the diagonal of the large square is  $4$ . Draw a square in the upper right corner with the bottom left vertex at the center of the circle. This square will have side lengths equal to the radius of the circle and a diagonal of  $\sqrt{2}r$ . The diagonal of the bottom left square will equal the radius of the circle  $r$  plus the  $\sqrt{2}r$ . This gives  $2 = r + \sqrt{2}r = r(1 + \sqrt{2})$ . The radius is  $r = 2/(1 + \sqrt{2})$  and the diameter is  $4/(1 + \sqrt{2})$ .  $\square$

15. On its natural domain, where is a local maximum of the function  $f(x) = x^3 - 3x^2$ ?

- (A) -2  
 (B) 0  
 (C) 1  
 (D) 2  
 (E) 4



SOLN  $f'(x) = 3x^2 - 6x = 3x(x - 2)$  so critical points are  $x = 0, 2$ .  $f''(x) = 6x - 6$  so the 2nd derivative test shows a local max at  $x = 0$  and local min at  $x = 2$ . The end behavior can be used instead of the 2nd derivative. □

17. The supremum of a set  $A$  of real numbers, notated as  $\sup A$ , is the smallest real number  $\geq$  to all the numbers in  $A$ . For instance,  $\sup([1, 2)) = 2$ . Analogously, we can define the infimum  $\inf([1, 2)) = 1$  for getting the greatest number  $\leq$  all the numbers in  $A$ . Let

$$A_N = \left\{ (-1)^n \cdot \left( 1 + \left( \frac{1}{2} \right)^n \right) \right\}_{n=N}^{\infty}$$

This means that  $A_N$  is all elements of the sequence starting at  $N$ . Then, calculate:

$$\lim_{N \rightarrow \infty} \sup A_N - \lim_{N \rightarrow \infty} \inf A_N$$

- (A) -1  
 (B) 0  
 (C) 1  
 (D) 2  
 (E) 3

SOLN The sequence bounces around between a sequence on the top that limits to 1 and a sequence on the bottom that limits to  $-1$ . In the process of limiting, the supremums decrease limiting to 1 and the infimums increase limiting to  $-1$ . So the difference of the two limits is 2. □

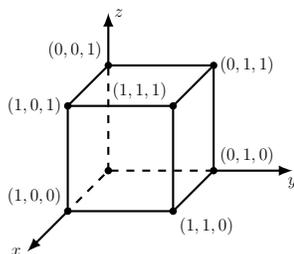
16. If  $f(x_1) + f(x_2) = f(x_1 + x_2)$  for all real numbers  $x_1$  and  $x_2$ , which of the following could define  $f$ ?

- (A)  $f(x) = x + 1$   
 (B)  $f(x) = 2x$   
 (C)  $f(x) = \frac{1}{x}$   
 (D)  $f(x) = e^x$   
 (E)  $f(x) = x^2$

SOLN This is called the Cauchy functional equation. Any function of the form  $f(x) = cx$  for some constant  $c$  solves this equation. Linearity is necessary, but this must also be true for  $x_1 = x_2 = 0$  which implies  $f(0) = 0$ . □

18. Imagine a 3-D unit cube tucked in the corner of the first octant; the vertices are ordered triples  $(0, 0, 0)$ ,  $(0, 0, 1)$ ,  $\dots$   $(1, 1, 1)$ . An edge is defined between any two vertices that differ in only one dimension. How many edges are there?

- (A) 8  
**(B) 12**  
 (C) 16  
 (D) 24  
 (E) 27



**SOLN** There are three ways to change  $(0, 0, 0)$  in one dimension, namely:  $(0, 0, 1)$ ,  $(0, 1, 0)$ , and  $(1, 0, 0)$ . There are two ways to additionally change each of those (that isn't back to the origin), but only three new points are achieved. From each of those three there is one final way to change to  $(1, 1, 1)$ .  $3 + 2(3) + 3 = 12$ .

Or, three edges emanate from each of the 8 vertices for a total of 24, but this counts each edge twice.  $\square$

19. Imagine a 4-D unit hypercube tucked in the corner of the first hyperoctant; the vertices are ordered quadruples  $(0, 0, 0, 0)$ ,  $(0, 0, 0, 1)$ ,  $\dots$   $(1, 1, 1, 1)$ . An edge is defined between any two vertices that differ in only one dimension. How many edges are there?

- (A) 16  
 (B) 20  
 (C) 24  
 (D) 28  
**(E) 32**

**SOLN** The number of edges on an  $n$ -dimensional cube is the dimension multiplied by half the number of vertices.  $4 \times \frac{1}{2}(16) = 32$ . Method 2: Four edges emanate from each of the 16 vertices for a total of 64, but this counts each edge twice. <https://www.math.brown.edu/tbanchof/Beyond3d/chapter4/section05.html>  $\square$

20. What is the output of the following Python program?

```
a = 1
b = 2
while a < 20:
    print(a)
    a = b
    b = a + b
```

- (A)** the powers of 2 less than 20  
 (B) the squares of numbers less than 20  
 (C) the triangular numbers less than 20  
 (D) the counting numbers less than 20  
 (E) the prime numbers less than 20

**SOLN** Manually run through the computational loop a few times. Doing so produces 1, 2, 4, 8, 16.  $\square$

21. The Pauli spin matrices  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$  appear in quantum mechanics. They are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where  $i^2 = -1$ . What is  $(\sigma_2)^2$ ?

- (A)  $\sigma_1$   
 (B)  $\sigma_2$   
 (C)  $\sigma_3$   
 (D)  $-\sigma_2$   
**(E)  $I$**

**SOLN**

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Note:  $(\sigma_k)^2 = I$  for  $k = 1, 2, 3$ ; that is, squaring each one gives the identity. These are self-invertible matrices, a.k.a. involutory matrices where  $A = A^{-1}$ .  $\square$

22. In a pair of fair six-sided dice, one has the regular number of pips on each face but the other is blank on all faces. Which numbers of pips can you put on the faces of the second die so the sample space of the sum of rolling both dice is  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$  and every outcome has equal probability?

- (A) 0, 2, 4, 6, 8, 10
- (B) 1, 2, 3, 4, 5, 6
- (C) 1, 2, 2, 3, 3, 4
- (D) 2, 2, 4, 4, 6, 6
- (E) 0, 0, 0, 6, 6, 6

**SOLN** There are  $6 \times 6 = 36$  possible outcomes for the pair of dice. For the 12 possible sums to have equal probability they must each be achievable 3 ways.

+	1	2	3	4	5	6
0	1	2	3	4	5	6
0	1	2	3	4	5	6
0	1	2	3	4	5	6
6	7	8	9	10	11	12
6	7	8	9	10	11	12
6	7	8	9	10	11	12

23. In modular arithmetic,  $a \equiv b \pmod{c}$  if  $a/c$  has a remainder  $b$ . You can add, subtract, and multiply, but not divide in modular arithmetic. However, you can use the multiplicative inverse (provided it exists). What is the multiplicative inverse of  $16 \pmod{19}$ ? In other words, find  $x$  such that  $x \times 16 \equiv 1 \pmod{19}$ .

- (A) 4
- (B) 6
- (C) 9
- (D) 13
- (E) 15

**SOLN**  $6 \times 16 = 96$  and  $96 = 5 \times 19 + 1$ , so  $96 \equiv 1 \pmod{19}$ . □

24. The price of a stock rose 20% on Monday, fell 10% on Tuesday, and increased by  $1/6$  on Wednesday. By what percent did the price rise from before the market opened on Monday to after the market closed on Wednesday?

- (A) 24
- (B) 26
- (C) 28
- (D) 30
- (E) 32

**SOLN** Let  $s$  be the initial price. Monday:  $1.2s$ ; Tuesday:  $1.2s - 0.1(1.2s) = 1.08s$ ; Wednesday:  $1.08s + \frac{1}{6}(1.08s) = 1.26s$ . □

25. The function  $y = f(x)$  has zeros  $-2$  and  $6$ . Find the zeros of  $y = -3f(2 - 2x)$ .

- (A)  $\{2, -2\}$
- (B)  $\{5, 1\}$
- (C)  $\{4, -1\}$
- (D)  $\{-1, -5\}$
- (E)  $\{1, -3\}$

**SOLN**  $2 - 2x$  must equal  $-2$  or  $6$ , so  $x = 2, -2$ . □

26. For  $b > c > 0$ , both  $x^2 + bx + 8$  and  $x^2 + cx + 8$  factor over the integers. Find  $b - c$ .

- (A) 1
- (B) 2
- (C) 3
- (D) 4
- (E) 5

**SOLN**  $(x + 8)(x + 1) \implies b = 9$   
 $(x + 4)(x + 2) \implies c = 6$  □

27. Which of the following is a solution to the system of nonlinear equations.

$$xy = -8$$

$$yz = -9$$

$$xz = 10$$

(A)  $\left(-\frac{4\sqrt{5}}{3}, -\frac{6\sqrt{5}}{5}, -\frac{3\sqrt{5}}{2}\right)$

(B)  $\left(\frac{4\sqrt{5}}{3}, -\frac{6\sqrt{5}}{5}, -\frac{3\sqrt{5}}{2}\right)$

(C)  $\left(-\frac{4\sqrt{5}}{3}, -\frac{6\sqrt{5}}{5}, \frac{3\sqrt{5}}{2}\right)$

(D)  $\left(-\frac{4\sqrt{5}}{3}, \frac{6\sqrt{5}}{5}, -\frac{3\sqrt{5}}{2}\right)$

(E)  $\left(\frac{4\sqrt{5}}{3}, \frac{6\sqrt{5}}{5}, \frac{3\sqrt{5}}{2}\right)$

*SOLN* Multiply all three to get  $(xy)(yz)(xz) = (-8)(-9)(10)$ , so  $(xyz)^2 = 720$ . Hence  $xyz = \pm 12\sqrt{5}$ .

Using  $xy = -8$ , we find  $z = \frac{xyz}{xy} = \pm \frac{3\sqrt{5}}{2}$ . For  $xyz = 12\sqrt{5}$ , we take  $z = -\frac{3\sqrt{5}}{2}$ .

From  $\frac{xy}{xz} = \frac{-8}{10}$ , we get  $\frac{y}{z} = -\frac{4}{5}$ , so  $y = -\frac{4}{5}z$ . Substituting  $z = -\frac{3\sqrt{5}}{2}$  gives  $y = \frac{6\sqrt{5}}{5}$ . Then  $x = \frac{-8}{y} = -\frac{20}{3\sqrt{5}} = -\frac{4\sqrt{5}}{3}$ .

Thus one solution is  $(x, y, z) = \left(-\frac{4\sqrt{5}}{3}, \frac{6\sqrt{5}}{5}, -\frac{3\sqrt{5}}{2}\right)$ . Changing all signs gives a second solution  $(x, y, z) = \left(\frac{4\sqrt{5}}{3}, -\frac{6\sqrt{5}}{5}, \frac{3\sqrt{5}}{2}\right)$ .

Alternate solution:

Not (A) because  $xy < 0$ .

Not (B) because  $yz < 0$ .

Not (C) because  $xz > 0$ .

Not (E) because  $xy < 0$ . □

28. The sound intensity level (in decibels) is given by  $\text{SIL} = (10 \text{ dB}) \log_{10} \left(\frac{I}{I_0}\right)$ , where  $I$  is the sound intensity and  $I_0$  is a reference intensity (both in  $\text{W}/\text{m}^2$ ). If a sound is 1000 times more intense than the reference level, how many decibels louder is it?

(A) 10 dB

(B) 20 dB

(C) 30 dB

(D) 40 dB

(E) 60 dB

*SOLN* Since  $10 \log_{10} \left(\frac{I}{I_0}\right)$  and the sound is 1000 times as intense as the reference, we have  $10 \log_{10}(1000) = 10(3) = 30$  dB. Note: a decibel is  $\frac{1}{10}$  of a bel which is named after the inventor of the telephone. □

29. Given the following equations and inequalities, where each letter ( $A, B, C, D$ , or  $F$ ) represents a real number, which letter has the largest value?

$$D + C = B + F$$

$$D + A > C + F$$

$$A < B < D$$

- (A)  $A$   
 (B)  $B$   
 (C)  $C$   
 (D)  $D$   
 (E)  $F$

**SOLN** Immediately,  $A$  and  $B$  can be eliminated due to the last inequality. Take the middle inequality and subtract the first equation from each side.

$$\begin{array}{r} D + A > C + F \\ -(D + C) = -(B + F) \\ \hline A - C > C - B \end{array}$$

This is equivalent to  $A + B > 2C$ . This suggests that  $C$  is not the largest either.

The equation gives  $C = B + F - D$ . Substituting this into the second inequality line gives  $D + A > (B + F - D) + F$  which reduces to  $2D + A > 2F + B$ . Since  $A < B$ ,  $2D + B > 2D + A > 2F + B$  or  $2D > 2F$  making  $D$  the largest.  $\square$

30. What is the probability that the product of the numbers rolled on three fair six-sided dice is prime?

- (A)  $1/36$   
 (B)  $1/24$   
 (C)  $1/16$   
 (D)  $1/12$   
 (E)  $1/8$

**SOLN** The only possible prime products are 2, 3, and 5. There are 9 possible rolls to produce these factors, so  $P(\text{rolling a prime}) = 9/6^3 = 1/24$ .

Alt soln: The options are permutations of (1, 1, prime).  $3(\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{2}) = \frac{1}{24}$ .  $\square$

31. A bridge charges 2-axled vehicles a \$5 toll and 3-axled vehicles an \$8 toll. In an hour the bridge collected \$741 from 120 vehicles. How much would the bridge have collected if tolls were \$1 higher for 2-axled and \$2 higher for 3-axled vehicles?

- (A) \$888  
 (B) \$908  
 (C) \$926  
 (D) \$934  
 (E) \$1012

**SOLN** The system  $x + y = 120$  and  $5x + 8y = 741$  has solution  $x = 73$ ,  $y = 47$ . After the increases  $6(73) + 10(47) = 908$ .  $\square$

32. In the  $5 \times 5$  grid shown, each cell contains one of the digits 1 to 5 so that each row and each column has exactly one of each digit. Find the entry in row 3, column 4.

- (A) 1  
 (B) 2  
 (C) 3  
 (D) 4  
 (E) 5

1	2	5	4	3
2	5	4	3	1
5	4	3	1	2
4	3	1	2	5
3	1	2	5	4

**SOLN** For the Sudoku fans! Play till you get it!  $\square$

33. What is the remainder when 10 000 000 000 000 003 is divided by 9?

- (A) 0  
 (B) 1  
 (C) 4  
 (D) 7  
 (E) 8

**SOLN**  $9999999999999999 \bmod 9 = 0$ .  
 $9999999999999999 + 4 \bmod 9 = 0 + 4$ .  $\square$

34. What is the last digit of  $3^{2026}$ ?

- (A) 3
- (B) 1
- (C) 7
- (D) 9
- (E) 5

*SOLEN*  $3^0 = 1, 3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$ , etc. The cycle of last digits for powers of 3 is  $\{1, 3, 9, 7\}$ .  $2026 \bmod 4 = 2$ . So the last digit  $3^{2026} =$  last digit of  $3^2 = 9$ .  $\square$

36. Let  $\phi(n)$  be the number of integers 1 through  $n$  that only share a common factor of 1 with  $n$ . For example,  $\phi(4) = 2$  counts the numbers 1 and 3 which do not share a common factor with 4. Find

$$\phi(122) + \phi(61) + \phi(2) + \phi(1)$$

- (A) 120
- (B) 121
- (C) 122
- (D) 123
- (E) 124

*SOLEN* The numbers 1 through 122 can be partitioned into disjoint categories which do not overlap:

- Not a multiple of 61, 2, or 122: counted by  $\phi(122)$
- Multiple of 2, but not of 61 or 122: counted by  $\phi\left(\frac{122}{2}\right) = \phi(61)$
- Multiple of 61 but not of 2 or 122: counted by  $\phi\left(\frac{122}{61}\right) = \phi(2)$
- Multiple of 122: counted by  $\phi\left(\frac{122}{122}\right) = \phi(1)$

Hence, the sum just adds the sizes of the non-overlapping partition pieces which accounts for each number 1 through 122 exactly one time.  $\square$

35. A square has a side length of 10. The midpoints of each side are connected to form a diamond (rotated square). What is the area of the diamond?

- (A) 25
- (B) 30
- (C) 40
- (D) 45
- (E) 50

*SOLEN* The diamond is composed of 4 right triangles which each have an area of  $25/2$ .  $4(25/2) = 50$ .  $\square$

37. Evaluate the sum:

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)$$

- (A)  $\frac{1}{4}$
- (B)  $\frac{1}{2}$
- (C) 1
- (D) 2
- (E) 4

*SOLN*

$$\begin{aligned} S_n &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \dots \pm \frac{1}{n+1} \\ &= 1 \pm \frac{1}{n+1} \end{aligned}$$

$$\lim_{n \rightarrow \infty} S_n = 1 \quad \square$$

38. In  $\triangle ABC$ , the three sides have lengths 4, 5, and 6. The sum of the cosines (not the cosine of the sums) of the largest and smallest angles of  $\triangle ABC$  is a rational number  $m/n$  in lowest terms. Find  $m+n$ .

- (A) 15 = 7 + 8
- (B) 17
- (C) 24
- (D) 37
- (E) 45

*SOLN* Let the smallest angle be  $\alpha$  and the largest  $\beta$ . Law of cosines gives:

$$\cos \alpha = \frac{5^2 + 6^2 - 4^2}{2 \cdot 5 \cdot 6} = \frac{25 + 36 - 16}{60} = \frac{3}{4}$$

$$\cos \beta = \frac{5^2 + 4^2 - 6^2}{2 \cdot 5 \cdot 4} = \frac{25 + 16 - 36}{40} = \frac{1}{8}$$

$$\cos \alpha + \cos \beta = \frac{3}{4} + \frac{1}{8} = \frac{7}{8} = \frac{m}{n} \quad \square$$

39. Let  $p$  and  $q$  be prime numbers such that  $p^2 - 2q^2 = 1$ . How many such pairs  $(p, q)$  exist?

- (A) 0
- (B) 1
- (C) 2
- (D) 11
- (E) infinitely many

*SOLN*

It is clear that  $p > q$  and that  $p$  must be odd because  $p^2 = 1 + 2q^2$ . This, in turn, implies  $q$  must be even because  $2q^2 = p^2 - 1 = (p+1)(p-1) = 4m$  for some integer  $m$  because  $(p-1)$  and  $(p+1)$  are consecutive even numbers, one of which must be divisible by 4. So  $q$  must be 2 (the only even prime), and  $p$  must be 3.  $(p, q) = (3, 2)$ .  $\square$

40. How many different ways can you orient a cube to fit into a cubic slot fitted for it?

- (A) 12
- (B) 16
- (C) 20
- (D) 24
- (E) 28

*SOLN*

Six faces could be up. Each situation has four rotations.  $6 \times 4 = 24$ .  $\square$