# Snow College Mathematics Contest

$\boxed{\text{key}}$

**April 5, 2011** **Senior Division: Grades 10-12** **Form: $\boxed{\text{T}}$**

Bubble in the single best choice for each question you choose to answer.

1. What is the minimum number of times the graph of a fifth degree polynomial must cross the $x$-axis?

$\boxed{\text{(A)}}$ 1

(B) 2

(C) 3

(D) 4

(E) 5

$\boxed{\text{SOLN}}$ A polynomial of odd degree must cross at least once, but the other roots may be in complex conjugate pairs (and not cross the $x$-axis). □

2. How many numbers from 1 to 2011 are divisible by either 20 or 11?

(A) 282

$\boxed{\text{(B)}}$ 273

(C) 291

(D) 275

(E) 279

$\boxed{\text{SOLN}}$ Note on notation: $\lfloor x \rfloor$ (called "floor") gives the largest integer less than or equal to $x$. Use the inclusion-exclusion principle. The number of multiples of 11 and 20 up to 2011 are $\lfloor 2011/11 \rfloor = 182$ and $\lfloor 2011/20 \rfloor = 100$, but this double-counts the multiples of both 11 and 20. There are $\lfloor 2011/(11+20) \rfloor = 9$ of those. $\therefore$ the answer is $182 + 100 - 9 = 273$. □

3. What is the sum of the first seven cubes, $1^3 + 2^3 + 3^3 + \ldots + 7^3$ ?

(A) $15^2$

(B) $21^2$

$\boxed{\text{(C)}}$ $28^2$

(D) $36^2$

(E) $45^2$

$\boxed{\text{SOLN}}$ One method of solution is brute force. A quicker way is start a table and see the pattern that the sum of the first $n$ cubes is the square of the sum of the first $n$ natural numbers, that is, $\Sigma_{i=1}^{n} i^3 = (\Sigma_{i=1}^{n} i)^2$. □

4. The continued fraction expression for $\delta$ (the "silver ratio") is

$$\delta = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

Find a closed form expression for $\delta$.

$\boxed{\text{(A)}}$ $1 + \sqrt{2}$

(B) $\frac{1-\sqrt{2}}{2}$

(C) $\frac{1+\sqrt{2}}{2}$

(D) $\frac{\sqrt{2}}{2}$

(E) $\frac{2}{\sqrt{2}}$

$\boxed{\text{SOLN}}$ See that the continued fraction expression is equivalent to $\delta = 2 + \frac{1}{\delta}$. Multiply by $\delta$ and rearrange to get $\delta^2 - 2\delta - 1 = 0$ and use the quadratic formula. The two solutions are $1 \pm \sqrt{2}$, but only the positive one equals the original continued fraction. □

5. The expression
$$\frac{x}{1-x-x^2}$$
is a *generating function* for a famous sequence. Find the sequence by looking at the coefficients of the long division.

(A) Harmonic sequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

(B) Primes: $2, 3, 5, 7, 11, 13, \ldots$

(C) Squares: $1, 2, 4, 9, 16, 25, \ldots$

$\boxed{\text{(D)}}$ Fibonacci sequence: $1, 1, 2, 3, 5, 8, \ldots$

(E) Triangular numbers: $1, 3, 6, 10, \ldots$

$\boxed{\text{SOLN}}$

$$\begin{array}{r l}
 & x + x^2 + 2x^3 + 3x^4 + \ldots \\
1-x-x^2 \,\big) & x \\
 & \underline{x - x^2 - x^3} \\
 & x^2 + x^3 \\
 & \underline{x^2 - x^3 - x^4} \\
 & 2x^3 + x^4 \\
 & \underline{2x^3 - 2x^4 - 2x^5} \\
 & 3x^4 + 2x^5
\end{array}$$

$\square$

6. The *cardinality* (measure of the number of elements) of a set $A$ is denoted $|A|$. Two sets have the same cardinality if there is a one-to-one correspondence between them. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}$ be the integers, $\mathbb{Q}$ be the rational numbers, $\mathbb{R}$ be the reals, and $\mathbb{C}$ be the complex numbers. Which statement is **not** true?

(A) $|\mathbb{R}| > |\mathbb{Z}|$

(B) $|\mathbb{Q}| = |\mathbb{N}|$

$\boxed{\text{(C)}}$ $|\mathbb{C}| > |\mathbb{R}|$

(D) $|\mathbb{N}| = |\mathbb{Z}|$

(E) $|\mathbb{C}| > |\mathbb{Q}|$

$\boxed{\text{SOLN}}$ $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}| = |\mathbb{C}|$ $\square$

7. What is the multiplicative inverse (reciprocal) of the complex number $a + b\,\mathrm{i}$?

(A) $a - b\,\mathrm{i}$

$\boxed{\text{(B)}}$ $\frac{a-b\,\mathrm{i}}{a^2+b^2}$

(C) $-a - b\,\mathrm{i}$

(D) $\frac{1}{a} + \frac{1}{b}\,\mathrm{i}$

(E) $a^2 - b^2\,\mathrm{i}$

$\boxed{\text{SOLN}}$ $\frac{1}{a+b\,\mathrm{i}}\left(\frac{a-b\,\mathrm{i}}{a-b\,\mathrm{i}}\right)$ $\square$

8. A popular dice game is called craps. In it you roll two standard six-sided dice and add the numbers showing on the top faces. What is the probability of rolling a sum of either 7 or 11 (called "throwing craps")?

(A) $\frac{2}{11}$

$\boxed{\text{(B)}}$ $\frac{2}{9}$

(C) $\frac{1}{6}$

(D) $\frac{7}{36}$

(E) $\frac{6}{7}$

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

$\boxed{\text{SOLN}}$ Since they are mutually exclusive, $P(7 \text{ or } 11) = P(7) + P(11) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$. $\square$

9. What is the value of $\mathrm{e}^{\mathrm{i}\pi} + 1$?

(A) $-1$

$\boxed{\text{(B)}}$ $0$

(C) $1$

(D) $\pi$

(E) $\sqrt{2}$

$\boxed{\text{SOLN}}$ The formula $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$ is cute because it contains five of the most important numbers. $\square$

10. Which of the following sets of data does **not** determine the relative shape of a triangle?

(A) the ratio of two sides and the included angle

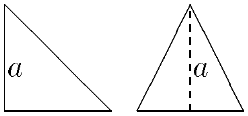
(B) the ratios of the three altitudes

(C) the ratios of the three medians

$\boxed{\text{(D)}}$ the ratio of the altitude to the corresponding base

(E) two angles

$\boxed{\textit{SOLN}}$



$\square$

11. Goldbach's conjecture (still an open question) says that every even integer greater than 2 is the sum of two primes. How many different ways can this be done for 24?

(A) 1

(B) 2

$\boxed{\text{(C)}}$ 3

(D) 4

(E) 5

$\boxed{\textit{SOLN}}$ $24 = 5 + 19 = 7 + 17 = 11 + 13$ $\square$

12. Simplify the expression for $\theta \neq n\pi$, $n \in \mathbb{Z}$.

$$\frac{\tan\theta - \sin\theta\cos\theta}{\tan\theta}$$

(A) $\sin\theta$

(B) $\cos\theta$

$\boxed{\text{(C)}}$ $\sin^2\theta$

(D) $\cos^2\theta$

(E) 1

$\boxed{\textit{SOLN}}$ $$1 - \frac{\sin\theta\cos\theta}{\frac{\sin\theta}{\cos\theta}} = 1 - \cos^2\theta$$ $\square$

13. What is the least number of prime factors (not necessarily different) that 350 must be multiplied by so that the product is a perfect cube?

(A) 1

(B) 2

(C) 3

(D) 4

$\boxed{\text{(E)}}$ 5

$\boxed{\textit{SOLN}}$ For a number to be a perfect cube each prime in the prime factorization must be cubed. $350 = 2 \cdot 5^2 \cdot 7$ so we need two more factors of 2, one more factor of 5, and two more of 7. $\square$

14. What is the equation of the line perpendicular to $y = -\frac{1}{2}x + 4$ and passes through through the point $(2, 6)$ ?

(A) $y = 2x + 10$

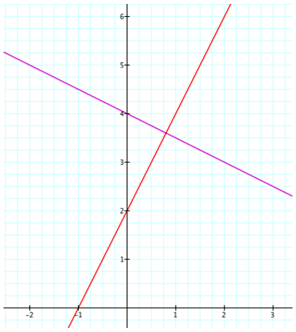
(B) $y = \frac{1}{2}x + 5$

(C) $y = x + 4$

$\boxed{\text{(D)}}$ $y = 2x + 2$

(E) $y = \frac{1}{2}x + \frac{1}{4}$

$\boxed{\textit{SOLN}}$



The slopes of perpendicular lines are negative reciprocals of each other. $\square$

15. How many equilateral triangles (of all sizes) are there in the figure?

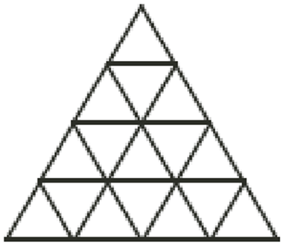
(A) 16

(B) 20

(C) 26

$\boxed{\text{(D)}}$ 27

(E) 32



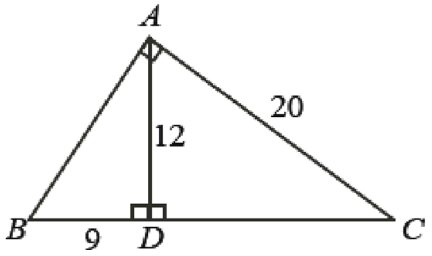
$\boxed{\textit{SOLN}}$ There are

16 (= 1+3+5+7) $1\times1\times1$ triangles,
7 (= 1+2+3+1) $2\times2\times2$ triangles,
3 $3\times3\times3$ triangles,
1 $4\times4\times4$ triangle

27 total $\square$

16. What is the perimeter of triangle $ABC$?

- $\boxed{\text{(A)}}$ 60
- (B) 56
- (C) 44
- (D) 54
- (E) 69



$\boxed{\text{SOLN}}$ $\triangle ABD$ is a right triangle, so $AB = 15$ (3-4-5 ratio or Pythagorean theorem). $\triangle ABC$ is a right triangle so $BC = 25$ (3-4-5 ratio or Pythagorean theorem). Therefore perimeter $P = AB + BC + CA = 15 + 25 + 20 = 60$. □

17. How many of the integers from 1 to 100 inclusive do **NOT** contain the digit 7?

- (A) 19
- (B) 20
- (C) 80
- $\boxed{\text{(D)}}$ 81
- (E) 90

$\boxed{\text{SOLN}}$ 1 through 100 is 100 numbers. There are 10 numbers ending in 7 (namely 7, 17, 27, …, 97). There are 10 numbers that start with 7 (namely 70, 71, 72, …, 79). However, 77 is in both sets so there are 19 numbers that have at least one 7. Therefore, there are $100-19 = 81$ numbers that do not contain the digit 7. □

18. What is the sum of the following?

$$2 - 1 + \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{8} - \tfrac{1}{16} + \dots$$

- $\boxed{\text{(A)}}$ $\frac{4}{3}$
- (B) $\frac{3}{2}$
- (C) 2
- (D) $\infty$
- (E) $\frac{7}{4}$

$\boxed{\text{SOLN}}$ This is a geometric series where $r = -\frac{1}{2}$. The series converges and

$$S = \frac{a_1}{1-r} = \frac{2}{1-(-\frac{1}{2})} = \frac{4}{3}$$

Cool solution #2 is to pair the terms:

$$(2-1) + (\tfrac{1}{2} - \tfrac{1}{4}) + (\tfrac{1}{8} - \tfrac{1}{16}) + \dots$$

$$= 1 + \tfrac{1}{4} + \tfrac{1}{16} + \dots$$

This is a geometric series with $r = 1/4$. The series converges and

$$S = \frac{a_1}{1-r} = \frac{1}{1-(\frac{1}{4})} = \frac{4}{3}$$

□

19. A rope hangs from the top of a pole with 3 ft of it lying on the ground. When it is tightly stretched so that its end just touches the ground the end is 8 ft from the base of the pole. How long is the rope?

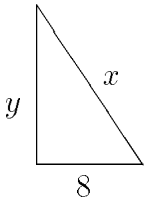
(A) 11 ft

(B) 12 ft

(C) 13 ft

(D) $\frac{55}{6}$ ft

$\boxed{\text{(E)}}$ $\frac{73}{6}$ ft

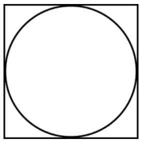


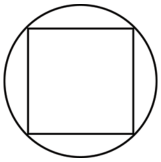
$\boxed{\text{SOLN}}$ Call the length of the rope $x$; we are given $y+3 = x$. From the triangle:

$$y^2 + 8^2 = x^2$$
$$(x-3)^2 + 8^2 = x^2$$
$$x^2 - 6x + 9 + 8^2 = x^2$$
$$6x = 73$$

This problem is from *Jiuzhang Suanshu*, an ancient Chinese math text. □

20. Which is a better (tighter) fit: a round peg in a square hole, or a square peg in a round hole? (A tighter fit fills up more of the hole.)





$\boxed{\text{(A)}}$ round peg in a square hole

(B) square peg in a round hole

(C) both are equally tight

(D) need to know the length of the side of the square

(E) need to know the radius of the circle

$\boxed{\text{SOLN}}$ The larger the ratio of the cross-sectional area of the peg to the area of the hole, the tighter the fit.

Round peg: $$\frac{\pi r^2}{(2r)^2} = \frac{\pi}{4}$$

Square peg: $$\frac{\left(\frac{2}{\sqrt{2}}r\right)^2}{\pi r^2} = \frac{2}{\pi}$$

$\frac{\pi}{4} > \frac{2}{\pi}$ because $\frac{\pi}{4} > \frac{3}{4} > \frac{2}{3} > \frac{2}{\pi}$ □

21. Ancient Greeks classified the brightest stars as first magnitude and so on until the dimmest they could see with the naked eye were sixth magnitude. Consider two stars, labeled 1 and 2, with apparent magnitudes $m_1$ and $m_2$ and brightnesses $b_1$ and $b_2$, respectively. The *ratio* of the apparent brightnesses $b_1/b_2$ corresponds to a *difference* in the apparent magnitudes $(m_2 - m_1)$.

$$m_2 - m_1 = 2.5 \log_{10}\left(\frac{b_1}{b_2}\right)$$

If $m_2 = 22$, $m_1 = 2$ how much brighter does star 1 appear than star 2; what is $b_1/b_2$?

$\boxed{\text{(A)}}$ $10^8$

(B) 2.5

(C) 400

(D) 4

(E) $\frac{1}{4}$

$\boxed{\textit{SOLN}}$

$$20 = \frac{5}{2}\log\left(\frac{b_1}{b_2}\right)$$

$$\left(\frac{2}{5}\right) 20 = \log\left(\frac{b_1}{b_2}\right)$$

$$8 = \log\left(\frac{b_1}{b_2}\right)$$

□

22. Evaluate: $(-125)^{-2/3}$

(A) $-25$

(B) $-\frac{1}{25}$

(C) 25

$\boxed{\text{(D)}}$ $\frac{1}{25}$

(E) $-\frac{1}{5}$

$\boxed{\textit{SOLN}}$ $\frac{1}{(-125)^{2/3}} = \frac{1}{(\sqrt[3]{-125})^2} = \frac{1}{(-5)^2}$ □

23. Say you buy 100 pounds of watermelon for a picnic. The melons are 99% water. By the date of the picnic, they dry out to 98% water. How much do they weigh now?

(A) 98 pounds

(B) 96 pounds

(C) 90 pounds

(D) 80 pounds

$\boxed{\text{(E)}}$ 50 pounds

$\boxed{\textit{SOLN}}$ When the watermelons weigh 100 lbs, 99 lbs are water and 1 lb is vegetable matter. You keep the 1 lb vegetable matter and let more water evaporate. When the lot weighs 50 lbs you have 49 lbs of water and 1 lb of vegetable matter. 49/50 = 98%. □

24. Flip a fair coin. Go 2 for heads, 1 for tails.

| | | | |
|---|---|---|---|
| Start | | Go back 2 spaces | End |

If the probability of reaching End in exactly 4 turns is $\frac{2}{16}$, in exactly 5 turns is $\frac{3}{32}$, and in exactly 6 turns is $\frac{5}{64}$, what is the probability of reaching End in exactly 7 turns?

$\boxed{\text{(A)}}$ $\frac{8}{128}$

(B) $\frac{5}{32}$

(C) $\frac{13}{256}$

(D) $\frac{1}{8}$

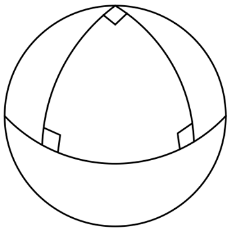
(E) $\frac{1}{4}$

$\boxed{\textit{SOLN}}$ The pattern is seen by computing the probability of reaching End in exactly 2 turns and exactly 3 turns. The probability of reaching End in exactly $n$ turns $(n \geq 2)$ is $\frac{F_{(n-1)}}{2^n}$ where $F_n$ is the $n$th Fibonacci number. □

25. The sum of the interior angles of a triangle on a sphere add up to more than $\pi$ rad by an amount $e$ called the *spherical excess.* The area of a spherical triangle is given by

$$A_\triangle = \frac{e}{4\pi} A_{\text{sphere}}$$

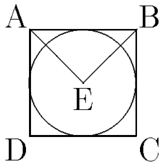
How much of a sphere does a spherical triangle with three right angles cover?



$\boxed{\text{(A)}}$ $\frac{1}{8}$
(B) $\frac{1}{4\pi}$
(C) $\frac{1}{4}$
(D) $\frac{3}{8}$
(E) $\frac{\pi}{4}$

$\boxed{\text{SOLN}}$ Three right angles sum to $\frac{3}{2}\pi$ rad so $e = \frac{\pi}{2}$. There are no similar (non-congruent) triangles on a sphere. $\square$

26. A circle of radius 2 and center E is inscribed inside square □ABCD. Find the area that is inside △AEB but outside the circle.



(A) $\pi - 3$
(B) $\frac{\pi}{2} - 1$
$\boxed{\text{(C)}}$ $4 - \pi$
(D) $\pi - 2$
(E) $3 - \frac{\pi}{2}$

$\boxed{\text{SOLN}}$ If $r = 2$ then the area of □ABCD is $4 \times 4 = 16$. The area of the circle is $\pi r^2 = \pi \cdot 2^2 = 4\pi$. △AEB represents 1/4 of both the square and the circle. desired area $= \frac{A_\square - A_\circ}{4} = \frac{16-4\pi}{4}$ $\square$

27. Of three boxes, one contains only apples, one contains only oranges, and one contains both apples and oranges. The boxes have been incorrectly labeled such that no label identifies the actual contents of the box it labels. Opening just one box, and without looking in the box, you take out one piece of fruit. By looking at the fruit, you can immediately label all of the boxes correctly. Which box do you open?

(A) the one labeled "apples"
(B) the one labeled "oranges"
$\boxed{\text{(C)}}$ the one labeled "apples and oranges"
(D) either "apples" or "oranges" will work
(E) any of the boxes will work

$\boxed{\text{SOLN}}$ Open the box labeled "apples and oranges"; if you pull out an apple that box must be only apples and then the one labeled "apples" must be oranges. Finally the one labeled "oranges" must be both. $\square$

28. Let $P(x) = x^3 - 2x^2 + 3x - 4$. Find the largest prime factor of $P(4) - P(2)$.

(A) 17
$\boxed{\text{(B)}}$ 19
(C) 23
(D) 29
(E) 31

$\boxed{\text{SOLN}}$ $P(4) = 4^3 - 2(4)^2 + 3(4) - 4 = 40$
$P(2) = 2^3 - 2(2)^2 + 3(2) - 4 = 2$
$P(4) - P(2) = 40 - 2 = 38 = 2 \cdot 19$ $\square$

29. The unique solution to $ax+b=10$ is $x=2$; the unique solution to $bx+a=8$ is $x=3$. Find $a+b$.

(A) $\frac{26}{5}$

$\boxed{\text{(B)}}$ $\frac{28}{5}$

(C) 6

(D) $\frac{32}{5}$

(E) $\frac{34}{5}$

$\boxed{\textit{SOLN}}$ Substitute the given $x$ into each equation. This gives a system of two equations in two unknowns ($a$ and $b$).

$$\begin{aligned} 2a + \phantom{3}b &= 10 \\ a + 3b &= \phantom{1}8 \end{aligned}$$

The solution is $a=\frac{22}{5}$ and $b=\frac{6}{5}$.
A slicker solution (which does not require finding $a$ and $b$ separately) is to multiply the first equation by 2 and then add the two equations.

$$5a+5b=28$$

□

30. Which polar equation describes the graph?

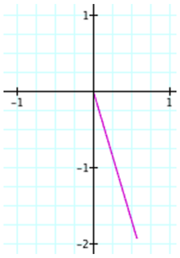
(A) $r=\theta$

(B) $r=5\pi$

(C) $r=\tan^{-1}\frac{y}{x}$

(D) $r=\sin 5\theta$

$\boxed{\text{(E)}}$ $\theta=5$



$\boxed{\textit{SOLN}}$ $\theta=c$ is a ray out from the origin. Here $\theta=5\,\text{rad}$ CCW from $+x$-axis. □

31. The Pauli spin matrices $\sigma_1$, $\sigma_2$, and $\sigma_3$ appear in quantum mechanics. They are

$$\sigma_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}\quad \sigma_2=\begin{bmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{bmatrix}\quad \sigma_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

What is $-\mathrm{i}\sigma_1\sigma_2\sigma_3$?

(A) 0

(B) $-1$

(C) i

(D) $-$i

$\boxed{\text{(E)}}$ $I$

$\boxed{\textit{SOLN}}$ Matrix multiplication is associative, but not commutative.

$$\sigma_2\sigma_3=\begin{bmatrix}0&\mathrm{i}\\\mathrm{i}&0\end{bmatrix}\qquad \sigma_1(\sigma_2\sigma_3)=\begin{bmatrix}\mathrm{i}&0\\0&\mathrm{i}\end{bmatrix}$$

$$-\mathrm{i}\sigma_1\sigma_2\sigma_3=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$$

Note: $I$, $\sigma_1$, $\sigma_2$, and $\sigma_3$ form a complete basis set for complex $2\times2$ matrices, so any matrix $A$ can be expressed as $A=c_0I+c_1\sigma_1+c_2\sigma_2+c_3\sigma_3$. □

32. For your vacation you will travel first to New York City, then to London. You may travel to NYC by car, train, bus, or plane, and from NYC to London by ship or plane. How many different routes are possible?

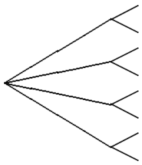
(A) 6

$\boxed{\text{(B)}}$ 8

(C) 10

(D) 12

(E) 14



$\boxed{\textit{SOLN}}$ Using a tree diagram we have four options for the first leg and two for the second leg, so there are 8 total. □

33. In Mrs. Austen's 3rd grade class there are 25 students total. Of those 25, 15 like Oreos, 9 like Fudge Stripes cookies, and 4 students don't like either. Determine the probability of choosing a student who likes both Oreos and Fudge Stripes cookies.

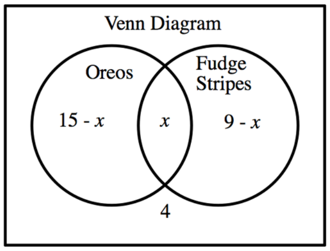
$\boxed{\text{(A)}}$ $\frac{3}{25}$

(B) $\frac{6}{25}$

(C) $\frac{19}{25}$

(D) $\frac{21}{25}$

(E) $\frac{24}{25}$



$\boxed{\text{SOLN}}$ $(15-x)+x+(9-x)+4=25 \implies 28-x=25 \implies x=3$ $\square$

34. What is the value of the following product?

$$\tan 5° \times \tan 15° \times \tan 25° \times \tan 35° \times \tan 45° \times \tan 55° \times \tan 65° \times \tan 75° \times \tan 85°$$

(A) $\frac{1}{2}$

(B) $\frac{\sqrt{3}}{3}$

(C) $\frac{\sqrt{3}}{2}$

$\boxed{\text{(D)}}$ 1

(E) $\frac{\sqrt{2}}{2}$

$\boxed{\text{SOLN}}$ $\tan(x)\tan(90-x)=1$ for $0<x<90$ $\square$

35. Let $R$ be the region below the curve $y=\sqrt{\sin(2x)}$ and above the $x$-axis with $0<x<\pi/2$. Find the volume of the shape generated by revolving $R$ around the $x$-axis.

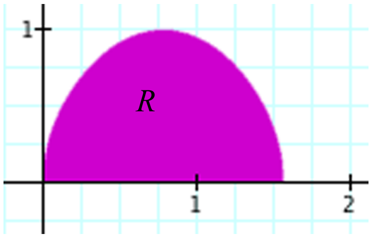
(A) 1

$\boxed{\text{(B)}}$ $\pi$

(C) $\frac{2\pi}{3}$

(D) 1

(E) $\frac{\sqrt{2}}{2}$



$\boxed{\text{SOLN}}$ Use the disk method. Each vertical disk has an area of $\pi(y)^2$. $\pi\int_0^{\pi/2}\sin(2x)\,\mathrm{d}x=\pi\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2}$ $\square$

36. All numbers in this question are in base four. What is $23^2$?

(A) 1121

(B) 1033

(C) 1031

(D) 2311

$\boxed{\text{(E)}}$ 1321

$\boxed{\text{SOLN}}$ Change to base ten: $23_{\text{four}}=11_{\text{ten}}$. Then $11^2=121$ and change back to base four: $121_{\text{ten}}=1321_{\text{four}}$. Or stay in base four:

$$\begin{array}{r} 23 \\ \times 23 \\ \hline 201 \\ 112\phantom{0} \\ \hline 1321 \end{array}$$

$\square$

37. At Chicken Littles you can order boxes of 6, 9, or 20 chicken fingers. What is the sum of the digits in the largest number of fingers you *cannot* order? (E.g., if 21 then $2+1=3$.)

(A) 5

$\boxed{\text{(B)}}$ $7=4+3$

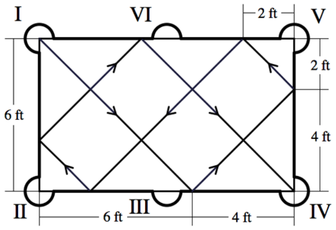
(C) 8

(D) 10

(E) 13

$\boxed{\text{SOLN}}$ With zero boxes of 20 (only boxes of 6 and 9) we can get any positive multiple of 3 (notated $3\mathbb{Z}^+$), *except* 3. With one box of 20 we can additionally get $20+3\mathbb{Z}^+$, *except* 23. This covers a second third of the positive integers (> 23). With two boxes of 20 we can additionally get $40+3\mathbb{Z}^+$, *except* 43. This covers the third third of the positive integers (> 43). They are all covered. Three boxes of 20 is a multiple of 6 and so covers no additional integers. It is important that 3 and 20 are relatively prime: $\gcd(3,20)=1$. $\square$

38. A large billiard table is 6 feet by 10 feet, and the cue ball moves as indicated. Into which hole will the cue ball fall?



(A) II

(B) III

$\boxed{\text{(C)}}$ IV

(D) I or V

(E) VI

$\boxed{\text{SOLN}}$ The law of reflection: $\theta_{\text{in}} = \theta_{\text{out}}$. Since the initial angle is 45° we will have isosceles right triangles all around. The first one will have sides of 6, the next will have sides of 4, then 2, then 8 (truncated), then 2, then 4, then 6, into pocket number IV.
N.B. (nota bene): real billiard tables are twice as long as they are wide. □

39. Jack can mow the lawn in 2 hours. His sister Jill can mow the same lawn in 3 hours. If they use two mowers and work together how fast can they mow the lawn?

(A) 1 hour and 10 minutes

(B) 1 hour and 30 minutes

(C) 1 hour and 15 minutes

$\boxed{\text{(D)}}$ 1 hour and 12 minutes

(E) 1 hour and 20 minutes

$\boxed{\text{SOLN}}$ Jack's rate of work is $\frac{1}{2}$ lawn/h and Jill's is $\frac{1}{3}$ lawn/h. The rate of both of them together is $(\frac{1}{2}+\frac{1}{3})$ lawn/h and the time it would take them together is the inverse of that.

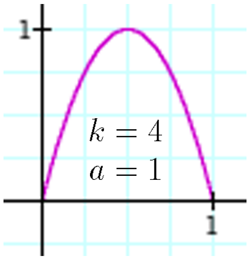
$$\frac{1}{\frac{1}{2}+\frac{1}{3}} = \frac{(2)(3)}{2+3} = \frac{6}{5}$$

$\frac{6}{5}\,\text{h} = 1\,\text{h}, 12\,\text{min}.$ □

40. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. If we assume that the rate of the reaction $v = \frac{dx}{dt}$ is proportional to both the amount of the product and to the amount of the original substance present then we write

$$v = kx(a - x)$$

where $x$ is the amount of product, $a$ is the amount of substance at the beginning, and $k$ is a positive constant. What is $v_{\max}$?



(A) $\frac{a}{2}$

(B) $kax - kx^2$

(C) $ka^2$

$\boxed{\text{(D)}}$ $\frac{ka^2}{4}$

(E) $\frac{k^2a}{2}$

$\boxed{\text{SOLN}}$ $v = kax - kx^2 \implies v' = ka - 2kx$
$v' = 0 \implies x = \frac{a}{2}$ $\quad v(\frac{a}{2}) = \frac{ka^2}{4}$
$v''(\frac{a}{2}) < 0 \implies$ maximum □